Gauge Theory as Cohomological Obstruction: A Structural and Constraint-Theoretic Formulation of Local Symmetry and Global Consistency

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Abstract

Gauge theory is traditionally formulated as a theory of local symmetry acting on field configurations. In this paper, we present a structural reformulation in which gauge systems are understood as local-to-global consistency problems governed by cohomological obstruction.

Local field data are defined over spacetime regions, and gauge transformations act as transition relations between overlapping regions. A globally consistent field configuration exists if and only if these local data satisfy compatibility conditions under composition.

Failure of these conditions produces cohomological obstruction, preventing descent to a globally consistent configuration. Within this framework, anomalies, topological defects, and nontrivial field configurations arise as manifestations of obstruction.

This establishes gauge theory as a constraint-driven system in which physical admissibility is determined by the vanishing of cohomological obstruction, providing a unified structural interpretation of symmetry, consistency, and physical realization.

Contents

1 Introduction

Gauge theory provides the fundamental framework for describing interactions in modern physics. Its defining feature is the presence of local symmetry transformations acting on field configurations.

Despite its standard formulation in terms of symmetry groups and differential geometry, the underlying structure of gauge theory can be understood more fundamentally as a local-to-global consistency problem.

In this paper, we reformulate gauge theory as a system in which:

- local field data are defined over spacetime regions
- gauge transformations define transition relations
- global configurations arise through compatibility
- failures of compatibility produce obstruction

This establishes a direct correspondence between gauge structure and cohomology.

2 Local Field Data

Let M be a spacetime manifold covered by open regions $\{U_i\}$.

On each region U_i, we define local field configurations:

$$\phi_i : U_i \to E$$

These local sections represent field data defined independently on each region.

3 Gauge Transformations as Transition Relations

On overlaps $U_i \cap U_j$, local configurations are related by gauge transformations:

$$\phi_i = g_{ij} \cdot \phi_j$$

where g_{ij} is a transformation valued in a symmetry group G.

These transformations serve as transition relations between local data.

4 Global Consistency and Descent

A global field configuration exists if local data can be consistently glued together.

This requires compatibility conditions on triple overlaps:

$$g_{ij} \circ g_{jk} = g_{ik}$$

Definition. A collection of local field data admits descent if these compatibility conditions are satisfied.

When descent holds, the local fields define a global configuration:

$$\phi : M \to E$$

5 Cohomological Obstruction

When compatibility conditions fail, global descent is obstructed.

Definition. A cohomological obstruction is the failure of local data and transition relations to assemble into a globally consistent structure.

Such obstructions correspond to nontrivial cohomology classes.

Interpretation. Cohomology measures the extent to which local consistency fails to extend globally.

6 Examples of Obstruction

Topological Defects

Configurations such as monopoles and vortices arise when global trivialization is impossible despite local consistency.

Instantons

Nontrivial global field configurations that cannot be continuously deformed into trivial ones represent cohomological classes.

Gauge Anomalies

At the quantum level, symmetry may fail to be preserved globally, even if it holds locally.

This corresponds to obstruction induced by quantization.

7 Constraint-Theoretic Interpretation

We define a transformation:

$$(\text{local fields}, \text{transition relations}) \longmapsto (\text{global configuration})$$

This transformation is valid if and only if compatibility conditions are satisfied.

- input: local field data
- constraint: compatibility of transition relations
- output: global field configuration

Observation. The transformation fails precisely when cohomological obstruction is present.

8 Structural Diagram

Input	Constraint Evaluation	Output
$\{\phi_i\}, \{g_{ij}\}$	$\Longrightarrow$	Global Field
Local Data	Compatibility Conditions	Global Structure

9 Relation to Physical Admissibility

Physical realizability requires global consistency of field configurations.

Principle. A field configuration is physically admissible if and only if it is free of cohomological obstruction.

Thus:

- admissible configurations correspond to trivial cohomology
- nontrivial cohomology corresponds to physical obstruction

10 Outlook and Applications

This formulation enables:

- detection of global inconsistency in gauge systems
- classification of topological phases
- verification systems for symmetry-consistent field configurations
- integration into compiler frameworks for constraint evaluation

11 Conclusion

Gauge theory can be understood as a local-to-global consistency system in which gauge transformations define transition relations between local field data.

Cohomological obstruction determines whether these local data can be assembled into a globally consistent configuration.

This establishes a structural interpretation in which symmetry, topology, and physical admissibility are unified under the principle of global consistency.

References

[1] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, 1983.

[2] M. Nakahara, *Geometry, Topology and Physics*, Taylor & Francis, 2003.

[3] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982.

[4] J. Baez and J. P. Muniain, *Gauge Fields, Knots and Gravity*, World Scientific, 1994.

[5] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser, 1993.

[6] M. F. Atiyah, *Geometry of Yang–Mills Fields*, Scuola Normale Superiore Pisa, 1979.

[7] D. S. Freed, *Classical Field Theory and Supersymmetry*, IAS Lecture Notes, available online.

[8] J. Giraud, *Cohomologie non abélienne*, Springer, 1971.

[9] J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.