

Gödelian Recursive Incompleteness

Dynamics, Extension, and Invariants of Formal Systems

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Abstract

We study repeated Gödelian extension of effectively presented arithmetical theories, distinguishing the intensional data of a presentation from its extensional deductive content. The basic progression $T_{n+1} = T_n + \sigma_{\mathbf{T}_n}$ is classical in spirit; our focus is the compatibility of an extension scheme with the identification of distinct presentations of the same theory.

We formulate a descent problem, prove a necessary and sufficient criterion for descent to deductive-equivalence classes, and define its presentation defect. Our concrete results construct an infinite, uniformly effective family of arithmetizations of precisely the same Peano-arithmetical axiom set whose associated Gödel sentences yield a strictly increasing chain of deductive extensions. The first two members give a simple one-step non-descent witness. Thus the one-step canonical Gödelian extension scheme fails to descend through unrestricted extensional identification of effective presentations. We also give a restricted positive invariance criterion based on internally verified proof translations. An exact one-step classification identifies provable equivalence of consistency statements as the invariant, and a consistency-spectrum theorem realizes an ordered family of such invariants within one fiber. For a fixed iterable successor-presentation policy, we construct the coarsest extension-stable refinement of deductive equivalence and identify its quotient with observable progressions. The infinite-chain theorem shows that a single fiber of the presentation-forgetting map can have an image of infinite order-theoretic height under one-step Gödelian extension. We further prove a genuine delayed-divergence theorem: under a fixed inherited-axiom successor policy, two effective presentations of Peano arithmetic agree both initially and after one extension but yield strictly different deductive theories after two extensions. We strengthen this to arbitrary finite delay: the finite-depth presentation equivalences form a strictly decreasing hierarchy, witnessed entirely within the fiber of Peano arithmetic. We derive a first-difference ultrametric with a non-isolated ordinary progression, give a Π_2^0 upper bound for full-trace equality, and extend arbitrary finite delay to sound, effectively presented arithmetic base theories. We further give an exact contextual-consistency certification criterion: one uniform arithmetical proof of equivalence across all one-sentence contexts is equivalent to uniform internal equivalence of provability predicates. This certificate is stable under equivalent axiom adjunction and entails invariance at every finite Gödelian stage; the engineered delayed-divergence witness fails this stronger test.

The construction is an explicit instance of the established presentation-sensitivity phenomena studied by Feferman, not a claim of priority for their discovery. Further results concern preservation of prior derivations, strict extension, and the first limit stage; questions about transfinite behavior and executable realization remain.

1 Introduction

Gödel's incompleteness theorems establish fundamental limitations on sufficiently expressive, effectively axiomatized formal systems. For an appropriate consistent theory F , there exist statements

in the language of F that are not decidable within F .

A natural response is to extend the theory.

If G_F denotes a Gödel sentence associated with F , one may form

$$F' = F + G_F.$$

The extension resolves the status of G_F within the enlarged theory. It does not, however, eliminate the phenomenon of incompleteness. Under the appropriate hypotheses, the enlarged theory possesses its own Gödel sentence.

This produces not merely an isolated extension but a progression

$$F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots.$$

The purpose of this paper is to study the mathematical structure of such progressions.

Our central question is:

What structure is induced when Gödelian incompleteness is treated as an iterated transformation of formal theories rather than solely as a property of an individual formal theory?

2 Preliminaries

We begin by fixing the class of formal theories to which the construction will apply.

Let F denote an effectively axiomatized formal theory capable of representing a sufficient fragment of arithmetic.

We write

$$F \vdash \varphi$$

when φ is provable in F , and

$$F \nvdash \varphi$$

otherwise.

For an appropriate theory F , let G_F denote a Gödel sentence associated with F .

Under the relevant consistency assumptions,

$$F \nvdash G_F.$$

Depending upon the precise formulation and hypotheses adopted, additional assumptions determine whether

$$F \nvdash \neg G_F$$

also follows.

These assumptions will be stated explicitly whenever required.

3 Gödelian Extension

We introduce the basic transformation studied throughout the paper.

Definition 3.1 (Gödelian extension). *Let F be a formal theory satisfying the required incompleteness hypotheses. Define a Gödelian extension of F by*

$$\mathcal{I}(F) = F + G_F.$$

We call $\mathcal{I}$ a Gödelian extension operator whenever the assignment $F \mapsto G_F$ has been fixed sufficiently to make the construction well-defined on the chosen class of theories.

The qualification in the preceding definition is important.

A theory need not possess a unique syntactic sentence that deserves to be called “the” Gödel sentence independently of choices of coding, provability predicate, presentation, and construction.

Accordingly, part of the problem is to determine the appropriate domain and equivalence relation under which $\mathcal{I}$ should be regarded as an operator.

4 Presented Formal Systems

The construction of a Gödel sentence depends not only upon the deductive consequences of a theory but also upon the manner in which the theory, its syntax, and its provability relation are represented.

This motivates distinguishing an abstract deductive theory from a specific effective presentation of that theory.

Definition 4.1 (Presented formal system). *A presented formal system is a tuple*

$$\mathbf{T} = (T, \text{Ax}_T, \text{Pr}_T, \gamma_T),$$

where:

- T is the deductive theory;
- Ax_T is an effective presentation of its axioms;
- $\text{Pr}_T(x)$ is an arithmetized provability predicate associated with that presentation;
- γ_T is an effective coding of the relevant syntactic objects and proof relations.

The distinction between T and $\mathbf{T}$ will be important. Different effective presentations may determine the same deductively closed theory while inducing different internal representations of syntax and provability.

Let

$$q(\mathbf{T}) = [T]$$

denote the deductive theory determined by $\mathbf{T}$, modulo a chosen notion of deductive equivalence.

The map

$$q : \mathbf{T} \mapsto [T]$$

forgets the intensional presentation data while retaining the extensional deductive content.

5 Gödelian Extension Schemes

The notation G_T can obscure the choices involved in constructing a Gödel sentence. We therefore make the choice of extension rule explicit.

Definition 5.1 (Gödelian extension scheme). *A Gödelian extension scheme σ assigns to each admissible presented formal system $\mathbf{T}$ a sentence*

$$\sigma_{\mathbf{T}}$$

constructed effectively from the presentation of $\mathbf{T}$ and satisfying specified incompleteness conditions. The corresponding extension operator is

$$\mathfrak{I}_{\sigma}(\mathbf{T}) = \mathbf{T}^+,$$

where the underlying deductive theory of $\mathbf{T}^+$ is

$$T^+ = T + \sigma_{\mathbf{T}}.$$

For the canonical Gödelian case we may take

$$\sigma_{\mathbf{T}} = G_{\mathbf{T}},$$

while other choices, including Rosser-style constructions or related independent-sentence schemes, may be considered separately.

Beginning with $\mathbf{T}_0$, define

$$\mathbf{T}_{n+1} = \mathfrak{I}_{\sigma}(\mathbf{T}_n).$$

The resulting sequence

$$\mathbf{T}_0 \longrightarrow \mathbf{T}_1 \longrightarrow \mathbf{T}_2 \longrightarrow \cdots$$

will be called the σ -progression generated by $\mathbf{T}_0$.

We write

$$\text{Orb}_{\sigma}(\mathbf{T}_0) = \{\mathbf{T}_n : n < \omega\}$$

for its finite-stage orbit.

Thus the object under investigation is determined not only by the initial theory but by the pair

$$(\mathbf{T}_0, \sigma).$$

6 Admissible Extensions and Consistency

The assertion that $T \not\vdash \sigma$ is not by itself sufficient to justify adding σ while retaining consistency: T might prove $\neg\sigma$. We therefore distinguish the syntactic condition of non-derivability from the two-sided condition needed for a consistent proper extension. Throughout the elementary results below, all theories use one fixed first-order language and one classical deductive calculus; $T + \varphi$ denotes the deductive closure of $T \cup \{\varphi\}$.

Proposition 6.1 (Consistency criterion for adjunction). *If T is a consistent theory, then*

$$T + \varphi \text{ is consistent} \iff T \not\vdash \neg\varphi.$$

If additionally $T \not\vdash \varphi$, then $T + \varphi$ is a strict extension of T .

Proof. By the deduction theorem and classical negation, a contradiction is derivable from $T \cup \{\varphi\}$ exactly when T proves $\neg\varphi$. If $T \not\vdash \varphi$ but $T + \varphi \vdash \varphi$, the consequence sets cannot coincide. $\square$

Remark 6.2 (Scope of the incompleteness hypothesis). For an ordinary Gödel sentence constructed using a standard arithmetized proof predicate, consistency suffices for its unprovability, but stronger hypotheses may be needed for the unprovability of its negation. Rosser constructions can provide independence under consistency with their own constructional hypotheses. No unqualified claim of consistency preservation is made for an arbitrary sentence-selection scheme.

7 Intensional and Extensional Dynamics

Recursive Gödelian extension naturally gives rise to two levels of description.

At the intensional level, the iteration acts upon presented formal systems:

$$\mathbf{T} \longmapsto \mathfrak{I}_\sigma(\mathbf{T}).$$

At the extensional level, one may ask whether this transformation induces a corresponding operation upon deductive equivalence classes:

$$[T] \longmapsto [T^+].$$

For such an induced transformation to exist, Gödelian extension must be compatible with the equivalence relation used to identify formal theories.

This produces the following fundamental problem.

Question 7.1 (Descent problem). *Does there exist a transformation*

$$\bar{\mathfrak{I}}_\sigma$$

on deductive equivalence classes such that

$$q \circ \mathfrak{I}_\sigma = \bar{\mathfrak{I}}_\sigma \circ q?$$

Equivalently, if

$$q(\mathbf{T}) = q(\mathbf{S}),$$

must it follow that

$$q(\mathfrak{I}_\sigma(\mathbf{T})) = q(\mathfrak{I}_\sigma(\mathbf{S}))?$$

If the answer is affirmative for a chosen class of presentations and extension schemes, then recursive Gödelian extension descends to the corresponding quotient space of theories.

If the answer is negative, the failure of descent measures the dependence of the recursive progression upon intensional presentation data.

This distinction motivates the study of two related dynamics:

intensional dynamics on presented theories

and

extensional dynamics on equivalence classes of theories.

A principal objective of the present work is to determine which features of recursive incompleteness survive passage from the first level to the second.

8 Descent Criterion and Presentation Defect

Fix a class $\mathcal{P}$ of admissible presentations on which the scheme gives a total map $\mathfrak{I}_\sigma : \mathcal{P} \rightarrow \mathcal{P}$. Let $\mathcal{Q} = q(\mathcal{P})$ be the quotient by the chosen deductive equivalence, so that $q : \mathcal{P} \twoheadrightarrow \mathcal{Q}$. The next result is an elementary fact about quotient maps; its role is to state exactly what must be proved for any particular Gödelian extension scheme.

Proposition 8.1 (Criterion for descent). *There exists a unique map $\bar{\mathfrak{I}}_\sigma : \mathcal{Q} \rightarrow \mathcal{Q}$ satisfying*

$$q \circ \mathfrak{I}_\sigma = \bar{\mathfrak{I}}_\sigma \circ q$$

if and only if, for every $\mathbf{T}, \mathbf{S} \in \mathcal{P}$,

$$q(\mathbf{T}) = q(\mathbf{S}) \implies q(\mathfrak{I}_\sigma(\mathbf{T})) = q(\mathfrak{I}_\sigma(\mathbf{S})).$$

Proof. The commuting identity immediately gives the stated implication. Conversely, define $\bar{\mathfrak{I}}_\sigma(q(\mathbf{T})) := q(\mathfrak{I}_\sigma(\mathbf{T}))$. The implication makes this definition independent of the chosen representative. Surjectivity of q makes the induced map unique. $\square$

Definition 8.2 (Presentation defect). *The defect of σ relative to q is*

$$D_\sigma(q) = \{(\mathbf{T}, \mathbf{S}) \in \mathcal{P}^2 : q(\mathbf{T}) = q(\mathbf{S}), \\ q(\mathfrak{I}_\sigma(\mathbf{T})) \neq q(\mathfrak{I}_\sigma(\mathbf{S}))\}.$$

Corollary 8.3 (Vanishing-defect test). *The extension scheme descends through q if and only if $D_\sigma(q) = \emptyset$.*

Proof. This is exactly the condition of Proposition 8.1. $\square$

Proposition 8.4 (A sufficient condition for descent). *Suppose q identifies presentations precisely when they determine the same deductively closed theory T in the same language. If for every pair $\mathbf{T}, \mathbf{S}$ in a single fiber of q their selected sentences satisfy*

$$T \vdash \sigma_{\mathbf{T}} \leftrightarrow \sigma_{\mathbf{S}},$$

then the extension scheme descends through q .

Proof. Provable equivalence over T makes the deductive closures $T + \sigma_{\mathbf{T}}$ and $T + \sigma_{\mathbf{S}}$ identical. Proposition 8.1 applies. $\square$

Remark 8.5 (Scope of the abstract criterion). Proposition 8.1 is a quotient-set criterion, not a new incompleteness theorem. The next section supplies a specified, effective, one-step non-descent witness. It also identifies a narrower, proof-translation-stable class of presentations on which invariance holds. Neither statement should be conflated with the claim that arbitrary standard changes of coding destroy invariance.

9 An Effective Non-Descent Witness for Gödelian Extension

We work throughout this section in classical first-order arithmetic with a fixed proof calculus and a fixed ordinary Gödel numbering. Write $\perp$ for $0 = 1$. Let $A(x)$ be a primitive-recursive axiom predicate for Peano arithmetic $P = \text{PA}$ (a standard effective coding of its axiom schemata). If $\chi(x)$

is a Σ_1 axiom predicate, write $\text{Prf}_\chi(p, x)$ for the usual witness-coded arithmetization of a finite proof using χ -axioms, $\text{Prov}_\chi(x) := \exists p \text{Prf}_\chi(p, x)$, and

$$\text{Con}_\chi := \neg \text{Prov}_\chi(\ulcorner \perp \urcorner).$$

The proof certificate p includes witnesses for any existential claims used to recognize a Σ_1 axiom. Consequently the certificate check is primitive recursive. All arithmetizations and proof transformations in this section use these witness-coded conventions. We assume externally that P is Σ_1 -sound; this supplies the required consistency and independence claims.

Put

$$U := P + \text{Con}_A, \quad C_U := \text{Con}(U), \quad \theta_U := \text{Prov}_U(\ulcorner \perp \urcorner).$$

Here Prov_U uses the ordinary recursively axiomatized presentation of U . Since P is Σ_1 -sound, U is consistent: otherwise P would prove the false Σ_1 sentence $\neg \text{Con}_A$.

Definition 9.1 (Consistency-triggered effective presentation). *Define two axiom predicates in the same language and numbering by*

$$\begin{aligned} \alpha(x) &:= A(x), \\ \beta(x) &:= A(x) \vee (x = \ulcorner \perp \urcorner \wedge \theta_U). \end{aligned}$$

Both are Σ_1 predicates (indeed their axiom sets are recursively enumerable). Let $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$ denote the resulting presentations, with their corresponding proof predicates and codings.

Operationally, the β -enumerator emits the usual axioms of P while searching for a U -proof of $\perp$. If that search ever succeeds, it additionally emits $\perp$ as an axiom. Because U is consistent, the extra axiom is never emitted in the standard model. This is a deliberately engineered presentation, not a mere renaming of Gödel codes.

Lemma 9.2 (Internal consistency calibration). *The presentations α and β have the same actual axioms, and P proves*

$$\text{Prov}_\beta(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_U(\ulcorner \perp \urcorner).$$

Consequently,

$$P \vdash \text{Con}_\alpha \leftrightarrow \text{Con}_A, \quad P \vdash \text{Con}_\beta \leftrightarrow C_U.$$

Proof. The actual axiom sets agree because the extra disjunct of β requires a U -proof of $\perp$, and no such standard proof exists.

For the displayed internal equivalence, first suppose a β -proof of $\perp$ is given, including witnesses for its Σ_1 -axiom tests. Examine its finitely many nonlogical axiom occurrences. If none uses the additional $\perp$ -disjunct, the same certificate is (after deleting the irrelevant witness tags) an α -proof of $\perp$. If an occurrence uses the additional disjunct, its certificate includes a U -proof of $\perp$. The case distinction and extraction of these certificates are primitive recursive; their correctness is formalizable in P . Thus

$$P \vdash \text{Prov}_\beta(\ulcorner \perp \urcorner) \rightarrow (\text{Prov}_\alpha(\ulcorner \perp \urcorner) \vee \text{Prov}_U(\ulcorner \perp \urcorner)).$$

Since U contains P , translating an α -proof to a U -proof is primitive recursive and verified in P . Therefore the right-hand side entails $\text{Prov}_U(\ulcorner \perp \urcorner)$.

Conversely, a U -proof of $\perp$ serves as a witness to θ_U , which licenses $\perp$ as a β -axiom; adjoining a one-line proof of that axiom is again a primitive-recursive, P -verifiable construction. Hence $P \vdash \text{Prov}_U(\ulcorner \perp \urcorner) \rightarrow \text{Prov}_\beta(\ulcorner \perp \urcorner)$. Negation gives the asserted equivalence of consistency sentences. The α equivalence holds by its definition. $\square$

Lemma 9.3 (Fixed-point calibration). *More generally, let χ be any witness-coded Σ_1 axiom predicate whose proof calculus and arithmetization satisfy the Hilbert–Bernays–Löb derivability conditions over P (as do the standard presentations used here, including their inherited-axiom successors). Let G_χ be any standard diagonal fixed point satisfying*

$$P \vdash G_\chi \leftrightarrow \neg \text{Prov}_\chi(\ulcorner G_\chi \urcorner).$$

Then

$$P \vdash G_\chi \leftrightarrow \text{Con}_\chi.$$

Proof. For the standard witness-coded predicates under consideration, these derivability conditions follow from the P -verified inclusion of P 's axioms in χ and elementary proof-certificate operations. First, each P -theorem has a χ -proof, so its χ -provability is provable in P . Second, primitive-recursive concatenation of witness-coded proofs formalizes closure of Prov_χ under modus ponens. Third, because Prov_χ is Σ_1 , formalized Σ_1 -completeness of P , together with the inclusion of its axioms among the χ -axioms, yields

$$P \vdash \text{Prov}_\chi(\ulcorner \varphi \urcorner) \rightarrow \text{Prov}_\chi(\ulcorner \text{Prov}_\chi(\ulcorner \varphi \urcorner) \urcorner)$$

for each sentence φ . These are the Hilbert–Bernays–Löb conditions needed below.

An encoded proof of $\perp$ can be converted to a proof of any sentence, so P proves $\text{Prov}_\chi(\ulcorner \perp \urcorner) \rightarrow \text{Prov}_\chi(\ulcorner G_\chi \urcorner)$. Together with the fixed-point equivalence, this gives $P \vdash G_\chi \rightarrow \text{Con}_\chi$.

In the reverse direction, the fixed-point equivalence proves in P that G_χ implies $\neg \text{Prov}_\chi(\ulcorner G_\chi \urcorner)$. By the first two derivability conditions, a χ -proof of G_χ therefore yields a χ -proof of this negated provability assertion. The third condition supplies a χ -proof of the positive provability assertion. Combining the two proofs yields a χ -proof of $\perp$. All these proof transformations are internal to P , so

$$P \vdash \text{Prov}_\chi(\ulcorner G_\chi \urcorner) \rightarrow \text{Prov}_\chi(\ulcorner \perp \urcorner).$$

Taking contrapositives and applying the fixed-point equivalence gives $P \vdash \text{Con}_\chi \rightarrow G_\chi$. □

Theorem 9.4 (Strict presentation-dependent Gödelian extension). *Assume $P = \text{PA}$ is Σ_1 -sound and use the same Gödel numbering and the same fixed-point construction on $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$ from Definition 9.1. Then:*

- (i) $q(\mathbf{P}_\alpha) = q(\mathbf{P}_\beta) = [P]$;
- (ii) both G_α and G_β are independent of P ;
- (iii) their Gödelian extensions satisfy the strict inclusion

$$\text{Th}(P + G_\alpha) \subsetneq \text{Th}(P + G_\beta).$$

In particular, for the scheme selecting an ordinary Gödel fixed point relative to each effective Σ_1 presentation,

$$(\mathbf{P}_\alpha, \mathbf{P}_\beta) \in D_\sigma(q), \quad D_\sigma(q) \neq \emptyset,$$

when q forgets the intensional axiom presentation and remembers only the deductively closed theory.

Proof. Part (i) is the extensional equality in Lemma 9.2. The two lemmas give the P -provable equivalences

$$G_\alpha \leftrightarrow \text{Con}_A, \quad G_\beta \leftrightarrow \text{Con}(U).$$

Thus $\text{Th}(P + G_\alpha) = \text{Th}(U)$ and $\text{Th}(P + G_\beta) = \text{Th}(P + \text{Con}(U))$. Because U extends P , the formalized inclusion of proof systems gives $P \vdash \text{Con}(U) \rightarrow \text{Con}_A$. It follows that the first deductive closure is included in the second. The inclusion is strict: if U proved $\text{Con}(U)$, it would violate Gödel's second incompleteness theorem, since U is a consistent recursively axiomatized extension of P . This proves (iii).

For (ii), P cannot prove G_α by the second incompleteness theorem for P , or G_β because a proof would also give $U \vdash \text{Con}(U)$. Neither negation is provable in P : the negations are P -provably equivalent to the false Σ_1 assertions $\neg \text{Con}_A$ and $\neg \text{Con}(U)$, respectively, and P is Σ_1 -sound. Therefore both are independent of P . Part (iii) supplies different images for presentations with the same q -image, which is exactly membership in the defect. $\square$

Corollary 9.5 (A formally detectable asymmetry). *For the two presentations in Theorem 9.4, P proves $\forall x(\text{Prov}_\alpha(x) \rightarrow \text{Prov}_\beta(x))$, but does not prove $\forall x(\text{Prov}_\beta(x) \rightarrow \text{Prov}_\alpha(x))$.*

Proof. Every α -axiom is a β -axiom, and the certificate-preserving inclusion of proofs is formalizable in P . If the reverse universal implication were provable in P , then, on specializing to $\ulcorner \perp \urcorner$ and contraposing, P would prove $\text{Con}_\alpha \rightarrow \text{Con}_\beta$. The calibrations would imply $U \vdash \text{Con}(U)$, contrary to Gödel's second incompleteness theorem. $\square$

Proposition 9.6 (Restricted invariance under certified proof equivalence). *Suppose χ and ψ are effective Σ_1 presentations of the same arithmetical theory, each admitting P -verified inclusion of P -proofs and the derivability conditions of Lemma 9.3. Let their fixed points be as above, and suppose P proves*

$$\forall x(\text{Prov}_\chi(x) \leftrightarrow \text{Prov}_\psi(x)).$$

Then $P \vdash G_\chi \leftrightarrow G_\psi$, so their one-step Gödelian extensions have identical deductive closures. In particular the hypothesis follows from P -verified uniform proof translations in both directions.

Proof. Specialization to $x = \ulcorner \perp \urcorner$ and negation show that P proves the equivalence of the associated consistency sentences. Lemma 9.3 identifies each fixed point with its consistency sentence, so $P \vdash G_\chi \leftrightarrow G_\psi$. Adding P -equivalent sentences to the same underlying theory yields the same deductive closure. Finally, P -verified total proof transformations in both directions give the stated uniform equivalence of provability. $\square$

Remark 9.7 (Historical and mathematical scope). The failure of invariance under unusual arithmetizations of one extensionally fixed theory is a known phenomenon in the literature, notably Feferman's analysis of numerations and consistency statements [2, 7, 3]. The theorem supplies an explicit witness for the descent framework of this manuscript and, unlike a schematic defect definition, proves an actual separation. The construction intentionally uses an intensional Σ_1 presentation that is not P -provably proof-equivalent to the ordinary one. It does *not* show that harmless or certified coding changes produce such a separation; Proposition 9.6 explains a restricted invariance principle.

10 Infinite Height Within One Presentation Fiber

The two presentations in Theorem 9.4 are the first members of a uniform family. The following observation calibrates one-step Gödelian extension against the consistency of an arbitrary recursively axiomatized strengthening of the fixed base theory.

Lemma 10.1 (Uniform consistency calibration). *Let $P = \text{PA}$ have its fixed ordinary axiom predicate $A(x)$, and let $V \supseteq P$ be a consistent recursively axiomatized theory, with a specified effective axiom enumeration. Set*

$$\chi_V(x) := A(x) \vee (x = \ulcorner \perp \urcorner \wedge \text{Prov}_V(\ulcorner \perp \urcorner)).$$

Then χ_V is an effective Σ_1 presentation of precisely the same actual axiom set as A , and its witness-coded proof predicate satisfies

$$P \vdash \text{Con}_{\chi_V} \leftrightarrow \text{Con}(V).$$

If G_{χ_V} is a diagonal fixed point for this proof predicate, then

$$P \vdash G_{\chi_V} \leftrightarrow \text{Con}(V).$$

The construction is uniform given an effective index for the axioms of V .

Proof. The additional disjunct is a Σ_1 statement because Prov_V is Σ_1 ; it can be enumerated by simultaneously enumerating the axioms of P and searching for a V -proof of $\perp$. Since V is consistent, the search never succeeds in the standard model, so the actual axioms remain those of P .

A χ_V -proof of $\perp$ either uses only ordinary P -axioms, in which case it can be translated into a V -proof, or uses the special $\perp$ -axiom, whose witness contains a V -proof of $\perp$. Conversely, any V -proof of $\perp$ witnesses the availability of that special axiom and hence gives a χ_V -proof of $\perp$. The case distinction and proof transformations are primitive recursive on witness-coded proof certificates, and P verifies their correctness. Thus

$$P \vdash \text{Prov}_{\chi_V}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_V(\ulcorner \perp \urcorner).$$

Taking negations gives the first displayed equivalence. The fixed-point equivalence follows from the derivability-condition argument of Lemma 9.3, which applies to every effective Σ_1 axiom presentation containing the ordinary axioms of P . All searches, enumerations, and certificate transformations are effective uniformly in an index for V . $\square$

Lemma 10.2 (Preservation of Σ_1 -soundness). *If V is Σ_1 -sound and π is a true Π_1 sentence, then $V + \pi$ is Σ_1 -sound.*

Proof. Suppose $V + \pi$ proved a false Σ_1 sentence φ . By the deduction theorem, V would prove $\pi \rightarrow \varphi$. But $\pi \rightarrow \varphi$ is equivalent in elementary arithmetic to the Σ_1 sentence $\neg\pi \vee \varphi$, which is false in the standard model. This contradicts Σ_1 -soundness of V . $\square$

Theorem 10.3 (Infinite height of a single presentation fiber). *Assume $P = \text{PA}$ is Σ_1 -sound. Define, using uniform effective presentations,*

$$V_0 = P, \quad V_{n+1} = V_n + \text{Con}(V_n) \quad (n < \omega).$$

For each n , set $\chi_n := \chi_{V_n}$ as in Lemma 10.1, let G_n be its ordinary diagonal Gödel fixed point, and let

$$E_n := \text{Th}(P + G_n).$$

Then:

- (i) *every χ_n is a uniformly effective Σ_1 presentation of exactly the ordinary axioms of P ;*
- (ii) *each G_n is independent of P and $P \vdash G_n \leftrightarrow \text{Con}(V_n)$;*

- (iii) *the images of these presentations under one-step Gödelian extension form a strictly increasing chain*

$$E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots$$

Consequently, the image of the single fiber $q^{-1}([P])$ under $q \circ \mathfrak{I}_\sigma$ contains a chain of order type ω . In particular, its range is infinite, and the presentation defect cannot be eliminated by treating the original counterexample as an isolated pair of presentations.

Proof. We first prove inductively that every V_n is Σ_1 -sound. This holds for $V_0 = P$ by hypothesis. If V_n is Σ_1 -sound, it is consistent; hence $\text{Con}(V_n)$ is a true Π_1 sentence. Lemma 10.2 gives Σ_1 -soundness of V_{n+1} . The theories V_n have uniform recursive axiomatizations by their recursive definition.

Lemma 10.1 now gives (i) and the stated equivalence in (ii). If P proved $\text{Con}(V_n)$, then V_n would prove its own consistency, contrary to Gödel's second incompleteness theorem for the consistent recursively axiomatized extension $V_n \supseteq P$. Nor does P prove its negation, since $\neg \text{Con}(V_n)$ is a false Σ_1 sentence and P is Σ_1 -sound. Thus every G_n is independent of P .

For (iii), the effective inclusion $V_n \subseteq V_{n+1}$ induces a P -verified proof translation. Therefore

$$P \vdash \text{Con}(V_{n+1}) \longrightarrow \text{Con}(V_n),$$

so $E_n \subseteq E_{n+1}$. Also, $E_n = \text{Th}(P + \text{Con}(V_n))$ is included in the deductive closure of $V_{n+1} = V_n + \text{Con}(V_n)$. If E_n proved $\text{Con}(V_{n+1})$, then V_{n+1} would prove its own standard consistency sentence, contradicting Gödel's second incompleteness theorem. Thus the inclusion is strict at every stage.

Finally, all χ_n lie over the same deductively closed base theory $[P]$, while their extensions have pairwise different deductive closures. This establishes the assertion about the image of the fiber. In this family, χ_0 is P 's ordinary presentation up to P -provable equivalence of consistency statements, and χ_1 reproduces the triggered-presentation construction of Theorem 9.4. $\square$

Remark 10.4 (Interpretation and prior art). The infinite chain arises from combining an established consistency progression with the presentation-calibration construction. It does not assert that the Feferman-style presentation dependence or the progression itself is new. The structural conclusion is stronger than the preceding single-pair witness: the one-step operation can have *infinite order-theoretic height on one fiber of the forgetful quotient*. This result concerns the variability of one-step images of a fixed base theory; it does not by itself prove divergent trajectories for repeated application of a single globally fixed presentation-selection rule.

11 Maximal Invariance Classes and the Consistency Spectrum

The previous sections exhibit failure of descent under unrestricted identification of effective presentations, as well as a sufficient invariance condition based on internally verified proof translations. We now identify an *exact* one-step criterion and the largest identification compatible with an entire iterated progression. These are different classification problems: agreement at the first successor stage does not, by itself, imply agreement at every stage.

11.1 An exact one-step classification

Fix a common arithmetical language, $P = \text{PA}$, and a deductively closed theory $T \supseteq P$. Consider effective axiom presentations χ, ψ of T , each with a witness-coded provability predicate containing the ordinary P -proof system in a P -verifiable way and satisfying the derivability hypotheses of

Lemma 9.3. Choose ordinary diagonal fixed points G_χ and G_ψ relative to these predicates. We use an axiom-predicate symbol also for its associated presented system when the intended object is clear. Write $\text{Th}(T + \rho)$ for the deductive closure of $T \cup \{\rho\}$.

Theorem 11.1 (Exact one-step invariance criterion). *Under the preceding hypotheses, the following conditions are equivalent:*

- (i) $\text{Th}(T + G_\chi) = \text{Th}(T + G_\psi)$;
- (ii) $T \vdash G_\chi \leftrightarrow G_\psi$;
- (iii) $T \vdash \text{Con}_\chi \leftrightarrow \text{Con}_\psi$.

Thus provable equivalence of the consistency assertions, not necessarily a verified translation between all proofs, is the exact invariant for the one-step extension generated by these fixed points.

Proof. Assume (i). Since G_ψ is an axiom of $T + G_\psi$, it is provable in $T + G_\chi$; by the deduction theorem, $T \vdash G_\chi \rightarrow G_\psi$. Interchanging χ and ψ yields the converse implication, proving (ii). Conversely, (ii) allows each added axiom to be derived from the other over T , so the two deductive closures coincide. Finally, Lemma 9.3 gives $P \vdash G_\chi \leftrightarrow \text{Con}_\chi$ and $P \vdash G_\psi \leftrightarrow \text{Con}_\psi$. Since T extends P , these equivalences establish (ii) $\Leftrightarrow$ (iii). $\square$

Let $\mathcal{P}$ denote the chosen set of presented systems and $q : \mathcal{P} \rightarrow \mathcal{Q}$ the map forgetting presentations. Define

$$\mathbf{T} \equiv_1 \mathbf{S} \iff q(\mathbf{T}) = q(\mathbf{S}) \text{ and } q(\mathfrak{J}_\sigma(\mathbf{T})) = q(\mathfrak{J}_\sigma(\mathbf{S})).$$

This is the largest equivalence contained in the kernel of q for which the *one-step observable extension* is independent of the chosen representative: any equivalence with both properties is a subset of $\equiv_1$. Theorem 11.1 describes its classes over $[T]$ by T -provable equivalence of consistency assertions, for the stated fixed-point scheme. The verified uniform proof translations of Proposition 9.6 are a sufficient certificate for membership; they are not part of the necessary-and-sufficient statement.

11.2 An order-theoretic realization of consistency strength

The presentation calibrations in Lemma 10.1 permit a stronger conclusion than infinite height alone. Let $\mathcal{V}$ be the collection of *indexed*, consistent, recursively axiomatized extensions $V \supseteq P$; each index includes a fixed effective axiom presentation and its associated standard consistency assertion. Define

$$V \preceq_{\text{Con}} W \iff P \vdash \text{Con}(W) \rightarrow \text{Con}(V),$$

and write $V \asymp_{\text{Con}} W$ when both directions hold. The resulting quotient $\mathcal{V} / \asymp_{\text{Con}}$ is a partially ordered set under $\preceq_{\text{Con}}$.

Theorem 11.2 (Consistency-spectrum representation). *Assume P is Σ_1 -sound. For every $V \in \mathcal{V}$ choose the consistency-triggered presentation χ_V of P from Lemma 10.1. Then*

$$\text{Th}(P + G_{\chi_V}) = \text{Th}(P + \text{Con}(V)).$$

Moreover, for all $V, W \in \mathcal{V}$,

$$\begin{aligned} V \preceq_{\text{Con}} W &\iff \text{Th}(P + G_{\chi_V}) \subseteq \text{Th}(P + G_{\chi_W}), \\ V \asymp_{\text{Con}} W &\iff \chi_V \equiv_1 \chi_W. \end{aligned}$$

Consequently the consistency-strength poset $\mathcal{V} / \asymp_{\text{Con}}$ embeds as an ordered family of one-step images within the single extensional fiber $q^{-1}([P])$.

Proof. Every χ_V enumerates exactly the ordinary axioms of P , since the consistency-triggering search never succeeds for a consistent V . Uniform calibration and the fixed-point lemma give $P \vdash G_{\chi_V} \leftrightarrow \text{Con}(V)$; the first displayed equality follows. Inclusion of the deductive closures is equivalent to the derivability $P + \text{Con}(W) \vdash \text{Con}(V)$, which by the deduction theorem is equivalent to $P \vdash \text{Con}(W) \rightarrow \text{Con}(V)$. Equality holds precisely when both implications are derivable. Because all χ_V lie above $[P]$, the definition of $\equiv_1$ then identifies exactly the pairs $V \asymp_{\text{Con}} W$. Quotienting by this equivalence makes the assignment injective and order-preserving and -reflecting. $\square$

Remark 11.3 (Scope of the spectrum). The theorem does not claim to classify all effective presentations of P : it gives an exact classification of the explicitly constructed family $\{\chi_V : V \in \mathcal{V}\}$ by their formalized consistency strengths. The infinite-chain theorem is one instance of this ordered realization. These are consistency assertions associated with *specified effective presentations* of V ; changing those presentations can change the assertions.

11.3 The largest extension-stable equivalence

To study an entire progression, fix a domain $\mathcal{P}$ of admissible presentations and an actually specified self-map $\mathfrak{I}_\sigma : \mathcal{P} \rightarrow \mathcal{P}$; in particular, the presentation of each successor theory must be selected, not just its underlying deductive closure. Assume the domain is closed under this operation. For $k \geq 0$ define

$$R_k = \{(\mathbf{T}, \mathbf{S}) : q(\mathfrak{I}_\sigma^j(\mathbf{T})) = q(\mathfrak{I}_\sigma^j(\mathbf{S})) \text{ for all } 0 \leq j \leq k\}, \quad R_\infty = \bigcap_{k < \omega} R_k.$$

A relation R is called *extension-stable* if $\mathbf{T} R \mathbf{S}$ implies $\mathfrak{I}_\sigma(\mathbf{T}) R \mathfrak{I}_\sigma(\mathbf{S})$.

Theorem 11.4 (Maximal extension-stable presentation equivalence). *The relation R_∞ has the following properties.*

- (i) *It is an extension-stable equivalence relation contained in $\ker q$.*
- (ii) *It is the largest such relation: if R is any extension-stable equivalence relation contained in $\ker q$, then $R \subseteq R_\infty$.*
- (iii) *The operator $\mathfrak{I}_\sigma$ induces a well-defined self-map on $\mathcal{P}/R_\infty$, and q factors through this quotient.*
- (iv) *Write*

$$\text{Tr}(\mathbf{T}) = (q(\mathbf{T}), q(\mathfrak{I}_\sigma \mathbf{T}), q(\mathfrak{I}_\sigma^2 \mathbf{T}), \dots) \in \mathcal{Q}^{\mathbb{N}}.$$

Then $\mathcal{P}/R_\infty$ is canonically in bijection with $\text{Tr}(\mathcal{P})$, and the induced extension operation corresponds to the left shift of these sequences.

Proof. Each R_k is an equivalence relation, being the intersection of kernels of functions $q \circ \mathfrak{I}_\sigma^j$. Their intersection R_∞ is therefore also an equivalence relation. Agreement at $j = 0$ establishes $R_\infty \subseteq \ker q$. If $(\mathbf{T}, \mathbf{S}) \in R_\infty$, then, for every $j \geq 0$,

$$q(\mathfrak{I}_\sigma^j(\mathfrak{I}_\sigma \mathbf{T})) = q(\mathfrak{I}_\sigma^{j+1} \mathbf{T}) = q(\mathfrak{I}_\sigma^{j+1} \mathbf{S}) = q(\mathfrak{I}_\sigma^j(\mathfrak{I}_\sigma \mathbf{S})),$$

proving extension stability and (i).

For (ii), let $R \subseteq \ker q$ be extension-stable. If $\mathbf{T} R \mathbf{S}$, induction on j gives $\mathfrak{I}_\sigma^j \mathbf{T} R \mathfrak{I}_\sigma^j \mathbf{S}$. The containment $R \subseteq \ker q$ implies equality of their q -images for every j . Thus $(\mathbf{T}, \mathbf{S}) \in R_\infty$, proving maximality.

For (iii), extension stability makes $[\mathbf{T}] \mapsto [\mathfrak{I}_\sigma \mathbf{T}]$ well-defined. The inclusion $R_\infty \subseteq \ker q$ makes $[\mathbf{T}] \mapsto q(\mathbf{T})$ well-defined. Finally, R_∞ is exactly the kernel of Tr , so $[\mathbf{T}] \mapsto \text{Tr}(\mathbf{T})$ gives the claimed canonical bijection. The identity $\text{Tr}(\mathfrak{I}_\sigma \mathbf{T}) = \text{shift}(\text{Tr}(\mathbf{T}))$ proves (iv). $\square$

Corollary 11.5 (Finite-depth refinement and universality). *The finite-depth equivalences obey*

$$R_0 = \ker q, \quad R_{k+1} = \ker q \cap (\mathcal{I}_\sigma \times \mathcal{I}_\sigma)^{-1}(R_k), \quad R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots,$$

and $R_\infty = \bigcap_k R_k$. *For any extension-stable equivalence relation $R \subseteq \ker q$, there is a canonical surjective map $\mathcal{P}/R \rightarrow \mathcal{P}/R_\infty$ commuting with the induced extensions and their q -observations.*

Proof. The recursive identity follows by separating the observation at $j = 0$ from those at $1, \dots, k+1$. The decreasing chain is immediate from the definitions. Theorem 11.4 gives $R \subseteq R_\infty$, so the assignment $[\mathbf{T}]_R \mapsto [\mathbf{T}]_{R_\infty}$ is a well-defined surjection. Both commutation statements follow from the defining quotient maps. $\square$

Corollary 11.6 (Maximal groupoid of admissible presentation changes). *Let $\mathcal{G}_q$ be the thin equivalence groupoid whose objects are the presentations in $\mathcal{P}$ and whose arrows $\mathbf{T} \rightarrow \mathbf{S}$ exist exactly when $q(\mathbf{T}) = q(\mathbf{S})$. The arrows satisfying $\mathbf{T} R_\infty \mathbf{S}$ form the largest wide subgroupoid $\mathcal{G}_\infty \subseteq \mathcal{G}_q$ on which $\mathcal{I}_\sigma$ acts as an endofunctor.*

Proof. Because R_∞ is an equivalence relation, it defines a wide subgroupoid closed under identities, inverses, and composition. Extension stability sends each arrow to an arrow and respects the unique compositions in this thin groupoid, so it defines an endofunctor. Conversely, the arrow relation of any wide subgroupoid of $\mathcal{G}_q$ preserved by the extension operator is an extension-stable equivalence relation contained in $\ker q$, hence lies in R_∞ by Theorem 11.4. $\square$

Corollary 11.7 (Infinitely many stable classes over Peano arithmetic). *For any total successor-presentation policy extending the one-step construction on the family $\{\chi_n : n < \omega\}$ from Theorem 10.3, the fiber $q^{-1}([P])$ contains infinitely many distinct R_∞ -classes.*

Proof. For distinct m, n , Theorem 10.3 yields $q(\mathcal{I}_\sigma \chi_m) \neq q(\mathcal{I}_\sigma \chi_n)$. Thus $(\chi_m, \chi_n) \notin R_1$ and, a fortiori, not in R_∞ . This uses only the first successor stage and is independent of the choices made at later stages. $\square$

Remark 11.8 (Why the one-step quotient is not the final answer). In a general deterministic system, R_1 need not be extension-stable. For example, suppose a, b have the same observation and their successors c, d also have the same observation, but the successors of c, d have different observations. Then $a R_1 b$ but $\mathcal{I}_\sigma a \not R_1 \mathcal{I}_\sigma b$. This is an abstract illustration, *not* a constructed example of these particular G ω delian schemes. It explains why the full-trace equivalence is needed for iterated invariance.

Remark 11.9 (Mathematical and historical scope). Theorem 11.1 characterizes one-step invariance for the stated class of arithmetizations; the consistency-spectrum theorem identifies a concrete ordered family inside one presentation fiber. The maximality theorem is a general result about deterministic systems with observations, closely related to standard behavioral equivalence and coalgebraic minimization [6]. Its application here distinguishes equality of one-step G ω delian extensions from equality of entire observed progressions. *Maximality does not imply decidability*: neither T -provability of the consistency biconditional nor equality of all future observations is claimed to be algorithmically decidable. Nor does the abstract theorem provide a natural successor-presentation policy automatically; that policy is part of the dynamical data.

12 Delayed Presentation Divergence Under an Inherited-Axiom Policy

The exact one-step criterion in Theorem 11.1 identifies when two presentations induce the same immediate extension. It does not entail that the resulting *presentations* have equivalent arithmetized proof predicates. This distinction matters when the presentation itself is inherited at the next stage. We now establish an explicit second-stage separation under a single fixed successor policy. In particular, the phenomenon occurs inside the same language and Gödel numbering, and without switching policies between stages.

Throughout this section let $P = \text{PA}$, use the witness-coded Σ_1 axiom and proof predicates fixed in Section 9, and put

$$C := \text{Con}(P), \quad U := P + C, \quad D := \text{Con}(U), \quad W := U + D.$$

The consistency assertions use the ordinary effective axiom presentations of P, U, W , respectively. We assume externally that P is sound in the standard natural numbers, so P, U, W are consistent. (For this explicit instance, this is the familiar soundness of the standard axioms and rules of PA .)

Definition 12.1 (Inherited-axiom successor policy). *Fix one effective diagonalization algorithm assigning to every admissible Σ_1 axiom presentation χ an ordinary Gödel fixed point G_χ for its associated provability predicate. Define*

$$S(\chi)(x) := \chi(x) \vee x = \ulcorner G_\chi \urcorner.$$

At every successor stage the old axiom enumerator is retained and the selected sentence is adjoined as one new axiom. The same rule and the same fixed-point algorithm are used at every stage.

Lemma 12.2 (Formalized consistency of a negative-consistency extension). *With $C = \text{Con}(P)$, the ordinary proof arithmetization satisfies*

$$P \vdash \text{Prov}_{P+\neg C}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_P(\ulcorner \perp \urcorner).$$

Equivalently, $P \vdash \text{Con}(P + \neg C) \leftrightarrow C$.

Proof. The formalized deduction theorem, with its primitive-recursive proof transformations, gives

$$P \vdash \text{Prov}_{P+\neg C}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_P(\ulcorner C \urcorner).$$

For the standard provability predicate $\Box\varphi := \text{Prov}_P(\ulcorner \varphi \urcorner)$, the formalized Löb theorem states $P \vdash \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$. Taking $\varphi = \perp$ yields

$$P \vdash \Box(\neg\Box\perp) \rightarrow \Box\perp,$$

which is exactly $P \vdash \text{Prov}_P(\ulcorner C \urcorner) \rightarrow \text{Prov}_P(\ulcorner \perp \urcorner)$. The reverse implication follows by formalized explosion. Combining the two implications proves the claim. $\square$

Definition 12.3 (A delayed consistency-triggered presentation). *Let $A(x)$ be the ordinary primitive-recursive axiom predicate of P . Put*

$$\Theta := \text{Prov}_W(\ulcorner \perp \urcorner), \quad \alpha(x) := A(x), \quad \beta(x) := A(x) \vee (x = \ulcorner \neg C \urcorner \wedge \Theta).$$

The predicates are effective Σ_1 axiom presentations with a common numbering and proof calculus. The β -enumerator emits the ordinary axioms of P and, while doing so, searches for a W -proof of $\perp$. Only if one is found does it emit $\neg C$.

Lemma 12.4 (Agreement at the first extension). *The presentations α and β have exactly the same actual axiom set, namely that of P . Moreover,*

$$P \vdash \text{Con}_\alpha \leftrightarrow \text{Con}_\beta,$$

and, for the fixed points in Definition 12.1,

$$\text{Th}(P + G_\alpha) = \text{Th}(P + G_\beta) = \text{Th}(U).$$

Proof. W is consistent, so Θ is false in the standard model, and the additional β -axiom never appears.

Every α -proof is a β -proof. Conversely, a witness-coded β -proof of $\perp$ either never uses the extra axiom, in which case it is a P -proof of $\perp$, or uses $\neg C$ as an axiom. In the latter case it can be transformed into a proof of $\perp$ from $P + \neg C$. The case distinction and proof transformations are primitive recursive and verifiable in P . Thus P proves

$$\text{Prov}_\beta(\ulcorner \perp \urcorner) \rightarrow \text{Prov}_{P+\neg C}(\ulcorner \perp \urcorner).$$

Lemma 12.2 converts the right side into $\text{Prov}_P(\ulcorner \perp \urcorner)$. The converse holds because β contains all P -axioms. Negation establishes $P \vdash \text{Con}_\beta \leftrightarrow C$; the same equivalence is immediate for α . Lemma 9.3 then yields

$$P \vdash G_\alpha \leftrightarrow C, \quad P \vdash G_\beta \leftrightarrow C.$$

Adding either sentence therefore yields exactly the deductive closure of $U = P + C$. $\square$

Lemma 12.5 (A concealed difference becomes visible at stage two). *Let $\alpha_1 := \mathbf{S}(\alpha)$ and $\beta_1 := \mathbf{S}(\beta)$. Although both present the same deductive theory U , their arithmetized consistency assertions satisfy*

$$P \vdash \text{Con}_{\alpha_1} \leftrightarrow \text{Con}(U), \quad P \vdash \text{Con}_{\beta_1} \leftrightarrow \text{Con}(W).$$

Proof. By the preceding lemma, P proves $G_\alpha \leftrightarrow C$ and $G_\beta \leftrightarrow C$. Uniform, P -verified substitution of these fixed proofs translates between proofs from $P + G_\alpha$, proofs from $P + G_\beta$, and proofs from $U = P + C$. Hence

$$P \vdash \text{Prov}_{\alpha_1}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_U(\ulcorner \perp \urcorner).$$

Now consider β_1 . A witness-coded β_1 -proof of $\perp$ either does not use the conditional axiom $\neg C$, or it does. In the first case, formalized translation yields a U -proof of $\perp$, and thus a W -proof of $\perp$ because $U \subseteq W$. In the second case, the proof certificate for $\neg C$ contains a witness to Θ , which is precisely a W -proof of $\perp$. These transformations yield

$$P \vdash \text{Prov}_{\beta_1}(\ulcorner \perp \urcorner) \rightarrow \text{Prov}_W(\ulcorner \perp \urcorner).$$

Conversely, from any W -proof of $\perp$ one obtains a witness to Θ , so that $\neg C$ becomes a licensed β_1 -axiom. Since $P \vdash G_\beta \rightarrow C$, the unchanged G_β -axiom and this licensed $\neg C$ give a β_1 -proof of $\perp$. The witness insertion and proof concatenation are primitive recursive and their correctness is verified in P . Thus

$$P \vdash \text{Prov}_W(\ulcorner \perp \urcorner) \rightarrow \text{Prov}_{\beta_1}(\ulcorner \perp \urcorner).$$

The two implications establish the desired equivalence of formalized inconsistency assertions; negating finishes the proof. $\square$

Theorem 12.6 (Delayed divergence of Gödelian presentations). *Under the fixed inherited-axiom successor policy S , there exist two effective Σ_1 presentations α, β of $P = \text{PA}$ in the same language and numbering such that*

$$\begin{aligned} q(\alpha) &= q(\beta) = [P], \\ q(S(\alpha)) &= q(S(\beta)) = [U], \\ q(S^2(\alpha)) &= [U + \text{Con}(U)] = [W], \\ q(S^2(\beta)) &= [U + \text{Con}(W)]. \end{aligned}$$

Moreover, the second-stage deductive closures are strictly nested:

$$\boxed{\text{Th}(W) \subsetneq \text{Th}(U + \text{Con}(W))}.$$

In the finite-depth notation of Section 11,

$$(\alpha, \beta) \in R_1 \setminus R_2.$$

In particular, R_1 is not extension-stable for this explicit Gödelian successor policy.

Proof. The equality at stage zero and stage one is Lemma 12.4. The successor presentations α_1 and β_1 contain P , so their standard diagonal fixed points satisfy

$$P \vdash G_{\alpha_1} \leftrightarrow \text{Con}_{\alpha_1}, \quad P \vdash G_{\beta_1} \leftrightarrow \text{Con}_{\beta_1}$$

by Lemma 9.3. The calibration in Lemma 12.5 and the stage-one equality then identify the second-stage underlying deductive theories as $W = U + \text{Con}(U)$ and $U + \text{Con}(W)$, respectively.

Because W extends U , the inclusion of proof systems is formalizable in P , giving $P \vdash \text{Con}(W) \rightarrow \text{Con}(U)$. Consequently $U + \text{Con}(W)$ proves every theorem of W . This inclusion is strict: if $W \vdash \text{Con}(W)$, it would contradict Gödel's second incompleteness theorem for the consistent, recursively axiomatized extension W of P . Hence the displayed strict inclusion holds.

Agreement at the first two observations means $(\alpha, \beta) \in R_1$; second-stage separation means $(\alpha, \beta) \notin R_2$. Finally, if R_1 were extension-stable, then $(S\alpha, S\beta) \in R_1$, contradicting the difference at their next observed stage. $\square$

Remark 12.7 (What the theorem establishes). The hidden axiom in β is deliberately designed; the theorem is an existence result, not a claim that routine changes of notation or Gödel coding cause delayed divergence. The successor policy, however, is the natural policy of retaining an axiom enumerator and adjoining one new fixed-point sentence at each stage. The result therefore distinguishes one-step and all-stages invariance in an explicit proof-theoretic setting, not merely in an abstract transition system. It uses classical formalized Löb reasoning and presentation-sensitive consistency. Priority or novelty relative to all earlier arithmetization work is not asserted.

13 Arbitrarily Delayed Presentation Distinguishability

Mathematical preamble: from an example to a depth theorem

The preceding section separates one-step equivalence from equivalence under all future extensions. The stronger question is whether this separation can be made to occur at *any prescribed finite depth*, while retaining the same diagonalization algorithm, axiom numbering, and inherited-axiom successor policy.

The answer is affirmative under the effective Σ_1 -presentation conventions already fixed above. The construction does not alter the actual axioms of the initial theory. Instead, it places one consistency-triggered axiom behind an arithmetically expressible condition. The consistency of that hidden axiom is formally invisible to sufficiently weak stages; after a selected consistency axiom has entered the progression, the same condition changes the next arithmetized consistency statement. This gives a concrete proof-theoretic separation at every finite observational depth.

Retain $P = \text{PA}$ and the fixed inherited-axiom policy $\mathbf{S}$ of Definition 12.1. Define the ordinary consistency progression

$$V_0 = P, \quad C_j := \text{Con}(V_j), \quad V_{j+1} := V_j + C_j \quad (j \geq 0),$$

using the standard uniformly effective presentations of the V_j . We assume externally that P is sound in the standard model, so all the V_j are consistent. For each fixed j , the ordinary provability predicate of V_j satisfies the derivability conditions, with their relevant proof transformations verifiable in P .

Lemma 13.1 (Consistency-neutral negative axioms below a boundary). *For all fixed integers $0 \leq j \leq k$,*

$$P \vdash \text{Prov}_{V_j + \neg C_k}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_{V_j}(\ulcorner \perp \urcorner).$$

Equivalently, $P \vdash \text{Con}(V_j + \neg C_k) \leftrightarrow C_j$.

Proof. Because V_k extends V_j , there is a primitive-recursive translation of V_j -proofs to V_k -proofs. Thus $P \vdash C_k \rightarrow C_j$. The formalized deduction theorem gives

$$P \vdash \text{Prov}_{V_j + \neg C_k}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_{V_j}(\ulcorner C_k \urcorner).$$

Formalized necessitation and modus ponens, applied to the fixed P -proof of $C_k \rightarrow C_j$, yield

$$P \vdash \text{Prov}_{V_j}(\ulcorner C_k \urcorner) \rightarrow \text{Prov}_{V_j}(\ulcorner C_j \urcorner).$$

The formalized Löb theorem for the ordinary provability predicate of V_j , applied to $\perp$, gives

$$P \vdash \text{Prov}_{V_j}(\ulcorner C_j \urcorner) \rightarrow \text{Prov}_{V_j}(\ulcorner \perp \urcorner).$$

Combining the displayed implications proves the forward direction. The converse follows by inclusion of V_j in $V_j + \neg C_k$, which is formally verified in P . $\square$

Definition 13.2 (A family of depth-controlled presentations). *For each $k \geq 0$ put*

$$H_k := V_{k+2}, \quad \Theta_k := \text{Prov}_{H_k}(\ulcorner \perp \urcorner),$$

and let $A(x)$ be the ordinary axiom predicate for P . Define

$$\alpha(x) := A(x), \quad \beta^{(k)}(x) := A(x) \vee (x = \ulcorner \neg C_k \urcorner \wedge \Theta_k).$$

The presentations use one common language, Gödel numbering, and proof calculus. The construction $k \mapsto \beta^{(k)}$ is uniform and effective. Since H_k is consistent, Θ_k is false in the standard model, so the two initial presentations have exactly the same actual axioms: those of P .

Lemma 13.3 (Formalized stage-by-stage calibration). *For the pair $\alpha, \beta^{(k)}$ of Definition 13.2, put $\alpha_j := \mathbf{S}^j(\alpha)$ and $\beta_j := \mathbf{S}^j(\beta^{(k)})$. Then, for $0 \leq j \leq k$,*

$$\begin{aligned} q(\alpha_j) &= q(\beta_j) = [V_j], \\ P \vdash \text{Con}_{\alpha_j} &\leftrightarrow C_j, \quad P \vdash \text{Con}_{\beta_j} \leftrightarrow C_j. \end{aligned}$$

At the next stage the presentations still have the same deductive content,

$$q(\alpha_{k+1}) = q(\beta_{k+1}) = [V_{k+1}],$$

but their formalized consistency statements differ:

$$P \vdash \text{Con}_{\alpha_{k+1}} \leftrightarrow C_{k+1}, \quad P \vdash \text{Con}_{\beta_{k+1}} \leftrightarrow C_{k+2}.$$

Proof. We spell out the formalization because this is the part on which iteration depends. Suppose the selected Gödel sentences at all prior stages $i < j$ have already been proved in P equivalent to C_i . By the fixed-point calibration lemma (Lemma 9.3), this follows whenever the corresponding formalized consistency statements are equivalent to C_i . Substituting these finitely many P -proved equivalences into proof certificates gives P -verified translations between proofs using the added Gödel-sentence axioms and proofs using the ordinary axioms $C_0, \dots, C_{j-1}$. These latter axioms present V_j .

For α_j this yields

$$P \vdash \text{Prov}_{\alpha_j}(\ulcorner \perp \urcorner) \leftrightarrow \text{Prov}_{V_j}(\ulcorner \perp \urcorner).$$

For β_j the original conditional axiom persists under the inherited-axiom policy. A witness-coded proof can be separated according to whether it uses that axiom; conversely a witness for Θ_k licenses each such use. Both directions are primitive-recursive operations on proof certificates, with correctness verifiable in P . Consequently,

$$P \vdash \text{Prov}_{\beta_j}(\ulcorner \perp \urcorner) \leftrightarrow (\text{Prov}_{V_j}(\ulcorner \perp \urcorner) \vee (\Theta_k \wedge \text{Prov}_{V_j + \neg C_k}(\ulcorner \perp \urcorner))). \quad (*)$$

For $j \leq k$, Lemma 13.1 reduces the right side of $(*)$ to $\text{Prov}_{V_j}(\ulcorner \perp \urcorner)$, regardless of Θ_k . Negation and the fixed-point calibration now give $P \vdash G_{\alpha_j} \leftrightarrow C_j$ and $P \vdash G_{\beta_j} \leftrightarrow C_j$. Adding these sentences produces the same deductive theory V_{j+1} and closes the induction. The base case $j = 0$ is the identical calculation with no previously adjoined Gödel sentences.

At $j = k + 1$, the accumulated explicit axioms already present V_{k+1} ; in particular they prove C_k . Thus there is a fixed, P -verifiably correct proof of $\perp$ from $V_{k+1} + \neg C_k$, and $(*)$ simplifies in P to

$$\text{Prov}_{\beta_{k+1}}(\ulcorner \perp \urcorner) \leftrightarrow (\text{Prov}_{V_{k+1}}(\ulcorner \perp \urcorner) \vee \Theta_k).$$

Since $H_k = V_{k+2}$ extends V_{k+1} , P proves that the first disjunct implies Θ_k . Hence the right side is provably $\Theta_k = \text{Prov}_{V_{k+2}}(\ulcorner \perp \urcorner)$. Negating gives the β_{k+1} consistency equivalence. The α_{k+1} equivalence follows from the preceding translations, and equality of deductive content at stage $k + 1$ was established by the induction. $\square$

Theorem 13.4 (Arbitrarily delayed Gödelian presentation divergence). *Fix the inherited-axiom successor policy $\mathcal{S}$ and the ordinary effective arithmetizations above. For every integer $d \geq 2$ there exist two effective Σ_1 axiom presentations $\alpha, \beta^{(d-2)}$ of exactly the same axiom set of $P = \text{PA}$ such that*

$$q(\mathcal{S}^j \alpha) = q(\mathcal{S}^j \beta^{(d-2)}) = [V_j] \quad (0 \leq j < d),$$

but at depth d ,

$$\begin{aligned} q(\mathcal{S}^d \alpha) &= [V_d], \\ q(\mathcal{S}^d \beta^{(d-2)}) &= [V_{d-1} + \text{Con}(V_d)], \end{aligned}$$

and the two deductive theories are strictly ordered:

$$\boxed{\text{Th}(V_d) \subsetneq \text{Th}(V_{d-1} + \text{Con}(V_d)).}$$

In particular, $(\alpha, \beta^{(d-2)}) \in R_{d-1} \setminus R_d$.

Proof. Set $k = d - 2$. The identities through stage $d - 1 = k + 1$ are Lemma 13.3. Its final consistency calibrations and Lemma 9.3 show that at stage $k + 1$ the two selected Gödel sentences are provably equivalent over P to C_{k+1} and C_{k+2} , respectively. The successor policy therefore yields the displayed stage- d theories, with $V_d = V_{d-1} + C_{d-1}$.

Because V_d extends V_{d-1} , the inclusion of the respective ordinary proof systems is verified in P , yielding $P \vdash C_d \rightarrow C_{d-1}$. Thus $V_{d-1} + C_d$ contains every theorem of V_d . The inclusion is strict: V_d is a consistent recursively axiomatized extension of P , so its ordinary formalized consistency assertion C_d is not provable in V_d , by Gödel's second incompleteness theorem. But C_d is an axiom of $V_{d-1} + C_d$. This establishes the strict containment and the claimed membership in $R_{d-1} \setminus R_d$. $\square$

Corollary 13.5 (Strictness at every finite observation depth). *For the fixed inherited-axiom scheme on the class of effective Σ_1 presentations of arithmetic, the finite-depth relations of Section 11 satisfy*

$$R_0 \supsetneq R_1 \supsetneq R_2 \supsetneq \cdots, \quad R_\infty = \bigcap_{d < \omega} R_d.$$

Furthermore, every strict inclusion is witnessed by a pair of presentations of exactly the same initial axiom set of P . In particular, no finite-depth relation R_d equals R_∞ on the entire presentation class.

Proof. Theorem 9.4 gives $R_0 \setminus R_1 \neq \emptyset$ within the fiber over P . For every $d \geq 2$, Theorem 13.4 gives $R_{d-1} \setminus R_d \neq \emptyset$ in the same fiber. The inclusions and the intersection identity are Corollary 11.5. If $R_d = R_\infty$ for a finite d , the strict inclusion $R_d \supsetneq R_{d+1}$ would be impossible. $\square$

Remark 13.6 (Scope and priority). The delay parameter controls *how many successor operations are needed to distinguish a particular pair*; it does not mean that the selected presentations disagree about any actual axiom of P . The presentations are engineered, recursively enumerable and extensionally identical, but generally not intensionally or provably interchangeable. Uniformity holds for constructing the pair from each finite d ; no algorithm is asserted for deciding R_d or R_∞ on arbitrary inputs. The result does not claim historical priority over existing work on arithmetized consistency, provability predicates, or recursive progressions; in particular Feferman's antecedents remain essential [2, 4].

14 Scope of Arbitrary Delay: Topology, Effectivity, and Transfer

The strictness of all finite-depth observational equivalences raises three distinct questions. Does it give a mathematical structure to the space of possible progressions? Does unbounded delay increase the computational complexity of comparing entire progressions? And is the phenomenon specific to Peano arithmetic, or can it be reproduced over other sound, effective arithmetic bases? We address these questions separately, retaining the same inherited-axiom successor policy.

14.1 A non-isolated progression in the trace space

Let $\mathcal{P}$ be the effectively presented arithmetic systems on which the fixed total successor operation S is defined, and let $\mathcal{X} = \mathcal{P}/R_\infty$ be the set of their complete extensional traces. For two distinct classes $[\chi], [\psi] \in \mathcal{X}$, define

$$m([\chi], [\psi]) := \min \{j \geq 0 : q(S^j \chi) \neq q(S^j \psi)\}, \quad \rho([\chi], [\psi]) := 2^{-m([\chi], [\psi])},$$

and put $\rho([\chi], [\chi]) = 0$. The definition is independent of representatives because R_∞ is equality of whole traces.

Proposition 14.1 (The first-difference ultrametric). *The function ρ is an ultrametric on $\mathcal{X}$. For every $k \geq 0$,*

$$(\chi, \psi) \in R_k \iff \rho([\chi], [\psi]) \leq 2^{-(k+1)}.$$

Thus the relations R_k are precisely the finite-prefix neighborhood relations in the observable trace space.

Proof. The minimum exists for distinct R_∞ -classes. If two traces agree through stage $a - 1$, and a second pair agrees through stage $b - 1$, then the first and third traces agree through stage $\min(a, b) - 1$. Consequently the first divergence index of the first and third traces is at least the minimum of the other two indices. Exponentiating with base $1/2$ gives the strong triangle inequality

$$\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}.$$

Symmetry, separation, and the displayed characterization of R_k follow directly from the definition of first disagreement. $\square$

Theorem 14.2 (The ordinary progression is not isolated). *Restrict $\mathcal{X}$ to complete traces arising from effective presentations of precisely the ordinary axiom set of $P = \text{PA}$. Let α denote the standard presentation in Definition 13.2. Then $[\alpha]$ is a non-isolated point of this trace subspace. In fact, there is a uniformly computable sequence of effective presentations $\beta^{(0)}, \beta^{(1)}, \dots$ of the same axiom set such that*

$$\rho([\alpha], [\beta^{(k)}]) = 2^{-(k+2)} \longrightarrow 0, \quad [\beta^{(k)}] \neq [\alpha] \quad (k \geq 0).$$

Proof. Theorem 13.4, applied with $d = k + 2$, identifies the first unequal deductive theory precisely at stage $k + 2$. Thus the first-difference metric has the stated value. Definition 13.2 makes the sequence effective, and its consistency trigger never fires in the standard model, so each presentation has exactly the ordinary axioms of P . The distances tend to zero while remaining positive, proving non-isolation. $\square$

Remark 14.3. Non-isolation is a topological expression of arbitrarily delayed observational distinguishability; it is not an additional claim of historical priority. The trace space of effective presentations is at most countable, since algorithms have countably many indices. No compactness, completeness, or continuum-cardinality claim is implied by the proposition.

14.2 A uniform arithmetical upper bound

Although equality of full infinite traces contains infinitely many conditions, their uniform effective generation permits a single arithmetical description. Assume the indexing of the admissible witness-coded presentations is effective and that S is a total computable transformation of indices. For a presentation index e , a stage n , and a finite certificate p , let $\text{Proof}(e, n, p)$ be the decidable assertion that p certifies a proof in the n -th successor presentation, and let $\text{End}(p)$ be its concluding sentence.

Proposition 14.4 (Arithmetical upper bound for full-trace equality). *Under the preceding effectiveness assumptions, the relation R_∞ on presentation indices is a Π_2^0 relation. The same upper bound applies to each R_k for fixed finite k . This is an upper bound, not a completeness or decidability result.*

Proof. The successor indices are computable uniformly from e, n ; proof certificates are decidable checkable. Inclusion of the n -th deductive theory of e in that of f is equivalent to

$$\forall p \exists r \left(\neg \text{Proof}(e, n, p) \vee [\text{Proof}(f, n, r) \wedge \text{End}(r) = \text{End}(p)] \right).$$

Equality of the theories imposes this condition in both directions. Equality at all finite stages adds a universal quantifier over n ; this is absorbed into the existing initial universal block. The two directions are combined by pairing their certificates. The resulting formula has a computable matrix and prenex form $\forall u \exists v$ with parameters e, f , proving the Π_2^0 upper bound. Restricting $n \leq k$ gives the assertion for R_k . $\square$

Remark 14.5. There is no contradiction between the strict chain $R_0 \supsetneq R_1 \supsetneq \dots$ and the common Π_2^0 upper bound. The hierarchy records *how many observations are needed*; the arithmetical hierarchy records the logical complexity of the global index relation. In particular, this proposition gives neither a computable finite stopping depth nor a decision procedure for R_∞ .

14.3 Transfer to a sound effective base theory

Theorem 14.6 (Base-theory transfer of arbitrary delay). *Let $B \supseteq P = \text{PA}$ be a sound, computably axiomatized arithmetical theory with a fixed witness-coded Σ_1 axiom presentation, and suppose its standard proof system includes that of P by a P -verified translation. Define*

$$V_0^B = B, \quad C_j^B = \text{Con}(V_j^B), \quad V_{j+1}^B = V_j^B + C_j^B.$$

Use the inherited-axiom successor policy of Definition 12.1 with the same effective fixed-point algorithm. For every $d \geq 2$ there exist effective Σ_1 presentations $\alpha_B, \beta_B^{(d-2)}$ of exactly the same axiom set of B whose first d extensional observations coincide, but whose depth- d extensions are strictly ordered:

$$\begin{aligned} q(S^j \alpha_B) &= q(S^j \beta_B^{(d-2)}) = [V_j^B] \quad (0 \leq j < d), \\ q(S^d \alpha_B) &= [V_d^B], \\ q(S^d \beta_B^{(d-2)}) &= [V_{d-1}^B + \text{Con}(V_d^B)], \end{aligned}$$

with

$$\text{Th}(V_d^B) \subsetneq \text{Th}(V_{d-1}^B + \text{Con}(V_d^B)).$$

The construction is uniform in the given axiom-presentation index of B and in d , as a construction on indices; soundness is a mathematical hypothesis and is not asserted to be decidable.

Proof. Fix $d = k + 2$. Let $A_B(x)$ denote the chosen axiom predicate, and use its fixed, effective coding together with the same proof calculus and fixed-point algorithm. Put

$$\Theta_k^B := \text{Prov}_{V_{k+2}^B}(\ulcorner \perp \urcorner), \quad \alpha_B(x) := A_B(x), \quad \beta_B^{(k)}(x) := A_B(x) \vee [x = \ulcorner \neg C_k^B \urcorner \wedge \Theta_k^B].$$

Because B is sound, every C_j^B is true, every V_j^B is sound, and the trigger Θ_k^B is false in the standard model. Hence α_B and $\beta_B^{(k)}$ enumerate exactly the same ordinary axioms of B .

The proof of Lemma 13.1 uses only formalized inclusion $V_j \subseteq V_k$, the deduction theorem, and Löb's theorem for ordinary provability predicates. These ingredients remain valid for V_j^B and are formalizable in P because the effective presentations uniformly include P . Thus, for $j \leq k$,

$$P \vdash \text{Con}(V_j^B + \neg C_k^B) \leftrightarrow C_j^B.$$

The witness-coded proof split in Lemma 13.3 now applies verbatim with V_j, C_j, Θ_k replaced by their B -indexed counterparts. It shows that through stage k the two arithmetized consistency assertions are P -provably equivalent to C_j^B , while at stage $k + 1$ they are P -provably equivalent to C_{k+1}^B and C_{k+2}^B , respectively. Fixed-point calibration yields the announced extensional stage identities.

Finally V_d^B extends V_{d-1}^B by the axiom C_{d-1}^B , and P verifies $C_d^B \rightarrow C_{d-1}^B$. Thus the right-hand theory contains V_d^B . Since the sound, effectively axiomatized theory V_d^B is consistent and contains P , Gödel's second incompleteness theorem gives $V_d^B \not\vdash C_d^B$, while the right-hand theory contains C_d^B as an axiom. The inclusion is strict. $\square$

14.4 A boundary at the first infinite stage

Proposition 14.7 (Extensional agreement at the ordinary limit). *Suppose two increasing finite-stage progressions have equal deductive theories at every stage $n < \omega$. Then their ordinary limit unions have the same deductive closure:*

$$\text{Th}\left(\bigcup_{n<\omega} T_n\right) = \text{Th}\left(\bigcup_{n<\omega} S_n\right).$$

In particular, R_∞ -equivalence guarantees extensional agreement at the union stage ω when this limit rule is used.

Proof. Any finite proof from the union uses finitely many axioms, each appearing at some finite stage. Since the sequences are increasing, all of these axioms occur together at one sufficiently large stage. A matching finite-stage deductive theory proves the same concluding sentence, so the corresponding union proves it. The reverse inclusion is symmetric. $\square$

Remark 14.8 (Limits of the present results). The proposition concerns the *deductive union at ω* . It does not show that the induced intensional presentations at that limit are equivalent, nor that their Gödel sentences at $\omega + 1$ agree; choosing an effective limit-stage axiom presentation is additional data. Likewise the finite-delay theorem does not yield a transfinite-delay theorem, a classification of all admissible natural presentations, or a historical-priority claim. The role of admissible numberings and stronger internal proof-translation requirements is discussed, for example, by Grabmayr [8]; Feferman's earlier results remain foundational [2, 4].

15 Contextual Consistency Certification and Iterated Invariance

Mathematical preamble: how strong must a certificate be?

The preceding results separate extensional equality of presentations from equality under repeated Gödelian extension. The delayed counterexamples rely on presentations with the same *actual* axioms but different arithmetized responses to an additional axiom. This prompts a certification question: which internal invariant is strong enough to survive every successor step, without requiring identical syntax, identical enumerators, or a chosen proof-by-proof translation as part of the definition?

There is no unqualified “weakest” certificate: the answer depends on which changes of context the certificate must tolerate. We therefore fix a natural and demanding interface: an arbitrary *single additional sentence*. In this interface the weakest requirement is exactly equality of the associated formalized consistency tests in *every* such context, established by one uniform proof in the base arithmetic $P = \text{PA}$. We show that this is equivalent to uniform internal equivalence of provability predicates and that it is preserved indefinitely by the inherited-axiom successor policy.

Throughout this section, χ, ψ are effective witness-coded Σ_1 axiom presentations in one arithmetical language and numbering. Their proof systems admit a P -verified embedding of P -proofs, uniform formalized deduction, and the ordinary Hilbert–Bernays–Löb derivability conditions, as in Lemma 9.3. Let $\text{Sent}(x)$ mean that x codes a closed formula, let $\text{Neg}(x)$ code its negation, and let $\chi + \theta$ denote the inherited-axiom presentation obtained by adjoining the fixed sentence

θ . Expressions with a variable context $\chi + \neg\varphi$ mean the arithmetically uniform family of such presentations. All displayed quantifiers over φ below abbreviate quantifiers over sentence codes; equality of codes is never being assumed.

Definition 15.1 (Uniform contextual-consistency certificate). *Write $\chi \approx_{\text{ctx}} \psi$ if there is a P -proof of*

$$\forall x (\text{Sent}(x) \rightarrow [\text{Con}_{\chi+\neg x} \leftrightarrow \text{Con}_{\psi+\neg x}]). \quad (\text{CC})$$

Here $\text{Con}_{\chi+\neg x}$ is the uniform formula asserting the absence of a proof of contradiction from χ plus the negation of the sentence coded by x . The certificate is a code of an ordinary P -proof of the single universally quantified formula (CC), not a list of external facts or a separate proof for each individual sentence.

Lemma 15.2 (Uniform formalized deduction). *For every such presentation χ , the fixed arithmetization admits P -verified, uniform proof conversions satisfying*

$$P \vdash \forall x (\text{Sent}(x) \rightarrow [\neg \text{Con}_{\chi+\neg x} \leftrightarrow \text{Prov}_\chi(x)]). \quad (\text{D})$$

For every fixed closed sentence θ , uniformly in the conclusion code x , it likewise satisfies

$$P \vdash \forall x (\text{Sent}(x) \rightarrow [\text{Prov}_{\chi+\theta}(x) \leftrightarrow \text{Prov}_\chi(\text{Imp}(\ulcorner \theta \urcorner, x))]), \quad (\text{D}^+)$$

where Imp is the computable operation coding implication.

Proof. For (D), a proof of contradiction using the extra axiom $\neg\varphi$ transforms, by the classical deduction theorem, into a χ -proof of $\neg\varphi \rightarrow \perp$, hence of φ . In the reverse direction, a χ -proof of φ combines with the newly admitted axiom $\neg\varphi$ to yield contradiction. All transformations operate uniformly on finite witness-coded proof objects and their correctness is provable in P . Formula (D⁺) is the same formalized deduction theorem with a fixed, arbitrary additional premise θ : remove its uses by implication introduction, or recover them by modus ponens from the new axiom. $\square$

Theorem 15.3 (Exact contextual certification criterion). *For the indicated class of presentations, the following are equivalent:*

- (i) $\chi \approx_{\text{ctx}} \psi$;
- (ii) P proves uniform equivalence of their entire formalized provability predicates:

$$P \vdash \forall x (\text{Sent}(x) \rightarrow [\text{Prov}_\chi(x) \leftrightarrow \text{Prov}_\psi(x)]). \quad (\text{PE})$$

Consequently (CC) is necessary and sufficient for provability-equivalence certification against every one-sentence consistency challenge. In particular, it is strictly a requirement about internal proofs; mere agreement of the two axiom sets in the standard model does not suffice.

Proof. For each coded sentence x , Lemma 15.2 identifies the negations of the two contextual consistency predicates with $\text{Prov}_\chi(x)$ and $\text{Prov}_\psi(x)$, respectively. The two equivalences are uniform in x and provable in P . Classical propositional logic therefore converts one P -proof of the universally quantified biconditional (CC) into one P -proof of (PE), and conversely. $\square$

Proposition 15.4 (Stability under equivalent axiom adjunction). *Suppose $\chi \approx_{\text{ctx}} \psi$ and θ, η are fixed closed sentences with $P \vdash \theta \leftrightarrow \eta$. Then*

$\chi + \theta \approx_{\text{ctx}} \psi + \eta.$

Proof. By Theorem 15.3, P proves uniform equivalence of Prov_χ and Prov_ψ . Applying this equivalence to the uniformly computable code $\text{Imp}(\ulcorner \theta \urcorner, x)$ shows that P proves

$$\text{Prov}_\chi(\text{Imp}(\ulcorner \theta \urcorner, x)) \leftrightarrow \text{Prov}_\psi(\text{Imp}(\ulcorner \theta \urcorner, x))$$

for all sentence codes x . The fixed P -proof of $\theta \leftrightarrow \eta$, combined with the P -verified embedding of P -proofs in ψ , gives a uniform verified conversion between ψ -proofs of $\theta \rightarrow \varphi$ and of $\eta \rightarrow \varphi$. Thus we may replace θ by η in the right-hand provability predicate. Formula (D⁺) in Lemma 15.2 identifies the outer terms with the predicates for $\chi + \theta$ and $\psi + \eta$. Another application of Theorem 15.3 proves the claim. $\square$

Theorem 15.5 (Indefinite certified invariance). *Fix the ordinary effective diagonal fixed-point selection and the single inherited-axiom successor policy S of Definition 12.1. If $\chi \approx_{\text{ctx}} \psi$, then*

$$S^n(\chi) \approx_{\text{ctx}} S^n(\psi) \quad \text{for every } n \in \mathbb{N}.$$

Consequently their deductive theories agree at every finite stage, so $\chi R_\infty \psi$ in the notation of Theorem 11.4. Moreover, from a proof of (CC) and the fixed presentation data one can effectively construct, for each n , a P -proof of the corresponding stage- n instance of (CC).

Proof. Induct on n . The case $n = 0$ is the assumption. Suppose $\chi_n := S^n(\chi)$ and $\psi_n := S^n(\psi)$ are contextually certified. By Theorem 15.3, P proves uniform provability equivalence, in particular $P \vdash \text{Con}_{\chi_n} \leftrightarrow \text{Con}_{\psi_n}$. Lemma 9.3 provides

$$P \vdash G_{\chi_n} \leftrightarrow \text{Con}_{\chi_n}, \quad P \vdash G_{\psi_n} \leftrightarrow \text{Con}_{\psi_n}.$$

Therefore $P \vdash G_{\chi_n} \leftrightarrow G_{\psi_n}$. Proposition 15.4, applied to these sentences, yields $\chi_n + G_{\chi_n} \approx_{\text{ctx}} \psi_n + G_{\psi_n}$, which is precisely the required successor statement. Theorem 15.3 gives equality of the deductive theories at each stage, because the actual provable sentences coincide. The proof constructions invoked in this induction—fixed-point calibration, formalized deduction, and insertion of a fixed derivation—are effective under the specified uniform arithmetization. Iterating them yields the claimed certificates at any given finite stage. $\square$

Corollary 15.6 (Maximality relative to the context interface). *Among equivalence relations on this presentation class whose members must be uniformly P -certified to give equal consistency outcomes under every one-sentence negative context, $\approx_{\text{ctx}}$ is the largest. It is also an extension-stable subrelation of R_∞ .*

Proof. Theorem 15.3 makes the required contextual condition exactly the defining property of $\approx_{\text{ctx}}$. As uniform provability equivalence is reflexive, symmetric, and transitive, it defines an equivalence relation. Any relation all of whose pairs satisfy (CC) is therefore a subrelation. Theorem 15.5 gives extension stability and containment in R_∞ . $\square$

Example 15.7 (A nontrivial certified change of presentation). Let A be the usual axiom predicate for P , and let τ be the logical validity $0 = 0$. The presentations $\chi(x) := A(x)$ and $\psi(x) := A(x) \vee [x = \ulcorner \tau \urcorner]$ are syntactically different. Nevertheless, any proof using τ as a separately listed axiom can uniformly replace that use by a fixed P -derivation of τ , and the reverse translation is immediate. These proof substitutions are verified in P . Thus $\chi \approx_{\text{ctx}} \psi$, and their canonical Gödelian progressions remain deductively equivalent at every finite stage by Theorem 15.5.

Proposition 15.8 (A dormant presentation fails contextual certification). *Let α, β be the presentations in Definition 12.3, and retain $C = \text{Con}(P)$, $U = P + C$, and $W = U + \text{Con}(U)$. Assume W is consistent. Then their first Gödelian extensions are deductively equal, but*

$$\alpha \not\sim_{\text{ctx}} \beta.$$

In fact the single extra context C distinguishes their formalized consistency statements in the sense that

$$P \vdash \text{Con}_{\alpha+C} \leftrightarrow \text{Con}(U), \quad P \vdash \text{Con}_{\beta+C} \leftrightarrow \text{Con}(W),$$

but $P \not\vdash \text{Con}(U) \leftrightarrow \text{Con}(W)$.

Proof. Lemma 12.4 gives $P \vdash G_\alpha \leftrightarrow C$ and $P \vdash G_\beta \leftrightarrow C$. Replacing those added axioms by C in the proof translations used in Lemma 12.5 yields the displayed formalized consistency equivalences for $\alpha + C$ and $\beta + C$. If P proved their target consistency assertions equivalent, then W , which proves $\text{Con}(U)$, would also prove $\text{Con}(W)$. This contradicts Gödel’s second incompleteness theorem for the consistent, recursively axiomatized extension W of P . Thus no uniform contextual certificate exists: such a certificate would in particular prove the biconditional for the context C (take the challenge formula $\neg C$ in (CC)). $\square$

Remark 15.9 (Precisely which “weakest” is established). One-step equality of Gödelian extensions requires only P -equivalence of the two *single* consistency statements (Theorem 11.1). For their one fixed infinite trajectory, the largest admissible equivalence is R_∞ ; no internal certificate for membership in R_∞ is asserted. The present theorem identifies instead the *weakest uniform internal condition expressed in the chosen one-sentence consistency challenge interface*. Under inherited-axiom extension it is a congruence and hence certifies the entire finite progression. It does *not* establish the absence of delayed divergence under all other proposed notions of “natural” presentation change, or settle the classification of acceptable Gödel numberings. The distinction between extensional correctness and internally adequate arithmetization has classical antecedents in Feferman [2]; restrictions on admissible numberings and formalized translation are also studied by Grabmayr [8]. The exact-context criterion and congruence proof are applications of the standard formalized deduction theorem, and are not asserted to be historically novel.

16 Proof Boundaries and Boundary Migration

Recursive extension also induces a natural stage-dependent notion of a proof boundary.

Let

$$B_n = \sigma_{\mathbf{T}_n}.$$

When B_n is independent of T_n under the hypotheses imposed on the chosen extension scheme,

$$T_n \not\vdash B_n.$$

After adjoining B_n , however,

$$T_{n+1} \vdash B_n.$$

At the same time, the strengthened system $\mathbf{T}_{n+1}$ generates a new boundary sentence

$$B_{n+1} = \sigma_{\mathbf{T}_{n+1}}.$$

Thus the progression has the schematic form

$$T_n \xrightarrow{B_n} T_{n+1},$$

with

$$T_n \not\vdash B_n, \quad T_{n+1} \vdash B_n, \quad T_{n+1} \not\vdash B_{n+1},$$

whenever the relevant independence hypotheses hold.

We call the resulting phenomenon *proof-boundary migration*: a specific limitation of the current theory is crossed by extension, while a new limitation appears relative to the strengthened theory.

The central structural question is not merely whether this process continues, but whether its evolution possesses properties invariant under changes of presentation, coding, or extension scheme.

17 Preservation, Strict Growth, and Warrant

An extension of a formal theory must be distinguished from replacing one physical model or interpretation with another. Adjoining axioms changes the set of derivable sentences while retaining all derivations from the earlier theory. Nothing in this assertion says that the physical world has changed, or that a formal theory has thereby become empirically more accurate.

Theorem 17.1 (Persistence and strict proof-boundary growth). *Let $(T_n)_{n<\omega}$ be a sequence of deductively closed theories in a fixed language and calculus such that $T_{n+1} = T_n + \sigma_n$. Then:*

- (i) *If $T_n \vdash \varphi$ and $m \geq n$, then $T_m \vdash \varphi$.*
- (ii) *If $T_n \not\vdash \sigma_n$, then $T_n \subsetneq T_{n+1}$.*
- (iii) *If, in addition, $T_n \not\vdash \neg\sigma_n$ and T_n is consistent, then T_{n+1} is consistent.*

Proof. Each extension contains every axiom and hence every theorem of its predecessor. Induction gives (i). The new sentence belongs to T_{n+1} but not to T_n , proving (ii). Claim (iii) follows from Proposition 6.1. $\square$

Corollary 17.2 (Persistent earlier warrant). *In the setting of Theorem 17.1, if $T_n \vdash \varphi$ then every later stage admits a formal proof of φ . In particular, the migration of a proof boundary cannot invalidate an earlier derivation merely by adding axioms in the same calculus.*

Definition 17.3 (Stage-indexed proof warrant). *For a presented theory $\mathbf{T}$ with an effective proof-checking relation $\text{Check}_{\mathbf{T}}$, write*

$$\text{Warr}_{\mathbf{T}}(\varphi) \iff \exists \pi \text{ Check}_{\mathbf{T}}(\pi, \ulcorner \varphi \urcorner) = 1.$$

Here a certificate π is a formal derivation, including whatever witnesses are needed to check the use of effectively enumerated axioms. Consequently $\text{Warr}_{\mathbf{T}}(\varphi)$ records syntactic provability, not an unconditional claim of truth in the standard model.

Remark 17.4 (Object-level and metatheoretic warrant). At stage n , the boundary sentence B_n has no proof warrant internal to T_n when $T_n \not\vdash B_n$. Its adjunction therefore requires a separate metatheoretic justification for the intended property of the extension (such as consistency or soundness), rather than a nonexistent T_n -proof of B_n . At stage $n+1$, the sentence has a derivation because it was adopted as an axiom. The status of a certificate, the status of an axiom, and truth in an intended interpretation must not be conflated.

Remark 17.5 (Extension versus empirical theory change). The monotonicity theorem concerns formal consequence under axiom adjunction; it is exact. When an empirical theory is superseded by a more accurate one, its established predictions may instead be recovered only in a regime or limiting approximation. A change in scientific representation does not by itself change the physical phenomena being represented. This comparison is motivational, not a claim that scientific theory replacement obeys the formal monotonicity theorem.

18 Finite Recursive Progressions

Having separated the presentation-dependent construction from its underlying deductive theory, we now examine the simplest finite Gödelian progression.

Fix a presented initial theory $\mathbf{T}_0$ and an extension scheme σ . When no confusion can arise, write F_n for the underlying deductive theory of $\mathbf{T}_n$ and G_{F_n} for the sentence selected by the canonical Gödelian scheme.

Define recursively

$$F_{n+1} = \mathcal{I}(F_n).$$

Thus,

$$F_1 = F_0 + G_{F_0},$$

$$F_2 = F_1 + G_{F_1},$$

and more generally

$$F_{n+1} = F_n + G_{F_n}.$$

This yields the sequence

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$$

At each stage the theory contains the preceding theory together with an additional sentence not previously derivable under the relevant hypotheses.

We call such a sequence a *Gödelian recursive progression*.

19 Incompleteness as Dynamics

The recursive construction permits a change in viewpoint.

Rather than regarding incompleteness only as a predicate

$$\text{Incomplete}(F),$$

we may study the transformation

$$F \mapsto \mathcal{I}(F).$$

Iteration then gives

$$F, \mathcal{I}(F), \mathcal{I}^2(F), \mathcal{I}^3(F), \dots$$

This suggests treating the class of admissible formal theories as a state space

$$\mathfrak{T},$$

with Gödelian extension acting schematically as

$$\mathcal{I} : \mathfrak{T} \longrightarrow \mathfrak{T}.$$

The analogy with a dynamical system must be treated carefully: $\mathfrak{T}$ need not initially possess a topology, metric, measure, or other conventional dynamical structure.

The first mathematical problem is therefore to determine what structure on $\mathfrak{T}$ is naturally induced by recursive extension.

20 Structural Invariants

Given

$$F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots,$$

we ask which properties remain invariant throughout the progression.

Candidate properties include:

- consistency;
- effective axiomatizability;
- interpretability of arithmetic;
- syntactic complexity;
- representability of provability;
- proof-theoretic strength;
- model-theoretic properties;
- computational properties.

An invariant J of the progression would satisfy an appropriate condition of the form

$$J(F_{n+1}) \simeq J(F_n),$$

where the meaning of $\simeq$ depends upon the structure represented by J .

Of equal interest are quantities that change monotonically or according to another identifiable law.

21 Branching Gödelian Extensions

The extension

$$F_{n+1} = F_n + G_{F_n}$$

represents only one possible progression.

Given a statement φ independent of F , one may consider

$$F + \varphi \quad \text{and} \quad F + \neg\varphi.$$

More generally, distinct choices of independent extensions may generate different paths through a space of theories.

This suggests replacing a single sequence by a branching structure

$$\begin{array}{ccc} & F & \\ / & & \backslash \\ F + \varphi & & F + \neg\varphi. \end{array}$$

The resulting structure may be more naturally represented by a partially ordered set, directed graph, category, or related object.

22 The Space of Formal Theories

Let

$$\mathfrak{T} = \{F : F \text{ satisfies the chosen admissibility conditions}\}.$$

A natural relation on $\mathfrak{T}$ is theory extension:

$$F \preceq F'$$

whenever F' extends F in an appropriate sense.

This produces a partially ordered structure under suitable identifications.

However, inclusion alone may discard important information.

Alternative relations include:

$$F \rightsquigarrow F'$$

for admissible extension,

$$F \rightsquigarrow F'$$

for interpretation, and

$$F \simeq F'$$

for an appropriate notion of equivalence.

The interaction between these relations will be investigated rather than fixed in advance.

23 Categorical Formulation

A richer description may be obtained by defining a category

Th

whose objects are admissible formal theories and whose morphisms encode appropriate transformations between theories.

Gödelian extension then becomes a candidate endomorphism-like construction

$$\mathcal{I} : \mathbf{Th} \longrightarrow \mathbf{Th},$$

provided suitable choices make the assignment functorial.

This raises several questions:

1. What should constitute a morphism between formal theories?
2. Under what conditions is Gödelian extension functorial?
3. Which properties are preserved under the resulting morphisms?
4. Can different Gödelian progressions be equivalent as categorical objects?
5. Is there a universal property associated with recursive extension?

24 Limit Stages

Finite iteration naturally suggests a limit construction.

For the first infinite stage, one may consider

$$F_\omega = \bigcup_{n < \omega} F_n.$$

This raises immediate questions.

Is F_ω effectively axiomatized?

Does it remain consistent if every F_n is consistent?

Does it satisfy the expressiveness hypotheses required for another application of incompleteness?

If so, can one form

$$F_{\omega+1} = F_\omega + G_{F_\omega}?$$

These questions lead naturally to transfinite progressions.

25 A First Limit-Stage Preservation Result

The first limit stage allows two elementary questions to be settled under explicit hypotheses, without presuming the availability of an arbitrary transfinite iteration.

Proposition 25.1 (Consistency and effectivity at ω). *Let $T_0 \subseteq T_1 \subseteq \dots$ be a chain of theories and let $T_\omega = \bigcup_{n < \omega} T_n$.*

- (i) *If every T_n is consistent, then T_ω is consistent.*

- (ii) *If there is a uniform effective procedure that, given n , enumerates axioms for T_n , then T_ω is effectively axiomatizable.*

Proof. An inconsistency would have a finite derivation using finitely many axioms from the union. All those axioms occur by some common finite stage, contrary to consistency at that stage. For (ii), dovetail the uniform axiom enumerations over n . This enumerates an axiom set for the union; its deductive closure is T_ω . $\square$

Remark 25.2. Uniformity in (ii) is essential; merely knowing that each member of an arbitrary sequence is individually effectively axiomatizable does not supply an effective enumeration of their union. At limit ordinals beyond ω , effectiveness additionally depends on the chosen system of ordinal notations and the way earlier stages are presented.

26 Transfinite Recursive Incompleteness

Let α range over suitable ordinals.

At successor stages define

$$F_{\alpha+1} = F_\alpha + G_{F_\alpha}.$$

At a limit ordinal λ , a candidate construction is

$$F_\lambda = \bigcup_{\beta < \lambda} F_\beta.$$

The resulting progression is

$$F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_\omega \rightarrow F_{\omega+1} \rightarrow \cdots.$$

However, the construction cannot simply be assumed to continue arbitrarily.

Effective presentation, consistency, ordinal notation, and the availability of an appropriate Gödel construction must be examined at each stage.

27 Fixed Points and Stabilization

A natural question is whether recursive extension admits any meaningful notion of stabilization.

A literal fixed point would satisfy

$$\mathcal{I}(F) = F.$$

For a theory satisfying the hypotheses under which G_F is independent, such equality appears incompatible with the construction.

Nevertheless, weaker notions may exist.

For example, one may ask whether there are theories for which

$$\mathcal{I}(F) \simeq F$$

under some structural equivalence weaker than equality.

Alternatively, certain invariants of the progression may stabilize even when the theories themselves do not.

This motivates the study of fixed points, periodicity, recurrence, and asymptotic invariants in an appropriately defined space of theories.

28 Recursive Incompleteness and Provability

The construction also changes the status of provability itself.

A sentence unprovable at stage n may become provable at stage $n + 1$:

$$F_n \not\vdash G_{F_n},$$

while

$$F_{n+1} \vdash G_{F_n}.$$

But F_{n+1} then possesses a new incompleteness phenomenon.

Thus provability is naturally indexed by the stage of the progression:

$$\text{Prov}_{F_0}, \text{Prov}_{F_1}, \text{Prov}_{F_2}, \dots$$

We investigate whether relationships among these provability predicates admit a unified structural description.

29 Computational Representation

Because the theories under consideration are effectively presented, one may ask whether finite segments of a Gödelian progression can be represented computationally.

A schematic architecture is

$$F_n \longrightarrow \ulcorner F_n \urcorner \longrightarrow G_{F_n} \longrightarrow F_{n+1}.$$

This raises a distinction between:

formal definability

and

effective executable construction.

Where the required encodings and transformations are effective, we may seek an executable representation of finite recursive progressions.

The limits imposed by incompleteness must remain explicit: computation does not remove the incompleteness phenomenon being studied.

Companion artifact and verification status. An exploratory software artifact, *Gödelian Contextual Certification—Lean 4 Starter*, accompanies this manuscript. It contains an abstract Lean 4 proof scaffold, a finite propositional experiment, and an inventory of outstanding proof obligations. The Lean source has not yet been compiled or machine-checked; its key arithmetical interfaces remain assumptions rather than proved theorems. The finite experiment does not verify Gödelian incompleteness results. The mathematical claims of this article therefore rest on the written proofs, with executable certification of the concrete arithmetical constructions reserved for future work.

30 Research Questions

The investigation is organized around the following questions:

1. What mathematical structure is induced by repeated Gödelian extension?
2. Which properties remain invariant along a recursive incompleteness progression?
3. Which properties increase, decrease, or otherwise evolve systematically?
4. How dependent is the progression upon the choice of Gödel sentence, coding, presentation, and provability predicate?
5. What structure is generated when multiple independent extensions are permitted?
6. Can the resulting space of formal theories be characterized order-theoretically or categorically?
7. Under what conditions can the progression be extended through limit ordinals?
8. Do meaningful fixed points or stabilization phenomena exist under weaker notions of equivalence?
9. Can finite portions of the progression be faithfully encoded, formally verified, and executed?
10. Does recursive incompleteness expose structural properties not visible when incompleteness is studied one theory at a time?

31 Relation to Existing Proof-Theoretic Progressions

Recursive extension of formal theories has important antecedents in proof theory, including transfinite progressions obtained by adjoining consistency or reflection principles.

Accordingly, the construction developed here must be compared carefully with established work on:

- Turing progressions,
- reflection principles,
- ordinal logics,
- consistency progressions,
- provability logic,
- ordinal analysis.

The objective is not to rename established constructions, but to determine whether the present formulation yields additional structure, new invariants, or new relationships among such progressions. The existence of Turing–Feferman progressions and the historical study of presentation and provability phenomena are antecedents, not discoveries claimed here [1, 2, 4, 5, 7]. The quotient criterion in Proposition 8.1 is abstract and elementary. Theorem 9.4 supplies a concrete non-descent witness using a deliberately engineered effective numeration, and Theorem 10.3 organizes such witnesses into an infinite strict chain; its purpose is to connect these established intensional phenomena rigorously

to the descent framework, not to assert a novel independence theorem. The exact one-step criterion in Theorem 11.1, the consistency-spectrum representation in Theorem 11.2, and the maximal stable quotient in Theorem 11.4 refine the classification question. The last construction is an instance of general behavioral equivalence for deterministic systems [6], not a new abstract minimization theorem. An explicit delayed-divergence example is provided in Section 12. Formal verification and additional restrictions on natural presentations remain research goals.

32 Conclusion

Gödelian incompleteness does not disappear when an undecidable sentence is incorporated into a stronger formal theory. Instead, under suitable conditions, incompleteness reappears relative to the enlarged system.

This observation motivates the study of incompleteness as an iterative phenomenon:

$$\boxed{F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_\alpha \longrightarrow \cdots}$$

The mathematical object of interest is therefore not only an incomplete formal theory, but the structure generated by repeated passage from one incomplete theory to another. The elementary results established here show that earlier derivations persist under axiom adjunction, that strict consistent growth has explicit proof-theoretic hypotheses, and that the first limit stage preserves consistency under an increasing chain. At the representational level, the presentation defect identifies exactly when a chosen extension scheme fails to define a transformation of deductive-equivalence classes.

Theorem 9.4 goes beyond the quotient criterion: under an explicit effective arithmetization, extensionally identical presentations of Peano arithmetic yield strictly nested Gödelian extensions. Thus a canonical fixed-point scheme relative to arbitrary effective Σ_1 presentations need not descend to the extensional quotient. Proposition 9.6 provides the corresponding positive result under internally verified proof equivalence. The example exhibits a known presentation-sensitivity phenomenon in the paper's descent notation; it does not claim priority over the literature on arithmetization. Theorem 10.3 further shows that a single fiber of the presentation-forgetting quotient has infinitely many strictly ordered one-step images, obtained uniformly from consistency progressions. Theorem 11.1 sharpens the positive criterion: for the fixed-point scheme under study, one-step invariance is equivalent to provable equivalence of the respective consistency statements. Theorem 11.2 realizes the preorder of consistency strengths, modulo provable equivalence, inside one presentation fiber. For an explicitly fixed iterated extension operator, Theorem 11.4 identifies the largest extension-stable refinement of deductive equivalence with equality of all observable stages; the corresponding quotient is a shift system of deductive-theory traces. These classification statements are not algorithmic decision procedures, and the abstract behavioral construction is already familiar from the mathematics of transition systems. Theorem 12.6 gives a concrete, second-stage separation under a single inherited-axiom successor policy: two presentations that agree initially and after one extension have strictly different second extensions. Theorem 13.4 strengthens this to every prescribed finite depth: the observational approximations R_d never stabilize at a finite stage, even within presentations of a single fixed axiom set. The next research steps are to assess robustness under more restrictive presentation equivalences and develop certified executable progressions. Section 15 resolves a precisely scoped certification problem: under all single-sentence challenges, uniform consistency equivalence is an exact criterion and a congruence for the inherited-axiom successor policy. Section 14 sharpens the established reach: the ordinary trace is not isolated in the first-difference ultrametric, full-trace index equivalence has a Π_2^0 upper bound, arbitrary delay

transfers to sound effective extensions of P , and an ordinary limit union preserves extensional agreement. None of these conclusions settles equivalence at a successor of a transfinite limit or establishes historical priority.

Section 15 supplies a positive boundary on presentation dependence: a uniform P -proof of contextual consistency equivalence is exactly equivalent to a uniform P -proof of predicate-level provability equivalence. Its stability under equivalent axiom adjunction gives a certificate for all finite successor stages, whereas the delayed witness is separated by a single counterfactual consistency context. This result identifies the maximal robust equivalence for the chosen challenge interface, not a unique weakest notion of mathematical certification in every possible setting.

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