Inevitability of Lie Algebras from Continuous Symmetry and Its Renormalization Group Analogue

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Abstract

Smooth associative composition induces structural rigidity, constraining subsequent evolution of the system. We prove that continuous symmetry with smooth associative composition admits a uniquely determined infinitesimal generator, and that closure under commutator forces a Lie algebra structure. In particular, the Lie bracket arises canonically from the commutator of invariant derivations, and the Jacobi identity is a consequence of associativity.

We identify an analogous rigidity principle for renormalization group flows: continuous coarse-graining transformations satisfying a semigroup law are necessarily generated by a beta vector field, whose linearization at fixed points governs phase structure.

In both settings, smooth composition eliminates independent global degrees of freedom and forces infinitesimal algebraic structure.

Contents

1 Introduction

Smooth one-parameter families of transformations arise throughout geometry and physics. Examples include group multiplication in Lie theory and scale evolution in renormalization group analysis. In each case, the family satisfies a composition law

$$F_{t+s} = F_t \circ F_s, \qquad F_0 = \mathrm{id}.$$

Such a family consists of infinitely many global transformations. However, the composition law together with smooth dependence on the parameter imposes strong structural constraints. The central question is:

> To what extent is a smooth one-parameter composition law determined by its infinitesimal data?

The main principle established here is that smooth associativity eliminates independent global degrees of freedom. Once smoothness and the composition law are imposed, the entire family is necessarily the flow of a uniquely determined vector field. Thus global composition is rigidly encoded by first-order structure.

We develop this principle abstractly for smooth composition structures and then specialize to two canonical settings:

- Lie groups, where the infinitesimal generator at the identity recovers the Lie algebra and defines the Lie functor;
- Renormalization semigroups, where the generator is the beta vector field governing scale evolution.

Finally, we express this mechanism categorically as a functor from smooth composition structures to vector fields, formalizing the passage from global composition laws to infinitesimal generators.

2 Continuous Composition and Infinitesimal Generators

We now isolate the structural mechanism underlying smooth one-parameter composition. A family $\{F_t\}_{t\in\mathbb{R}}$ consists of infinitely many global transformations, yet associativity together with smooth dependence on the parameter imposes strong rigidity. What appears as global freedom is in fact tightly constrained. We formalize this setting abstractly.

Definition 2.1. *Let X be a smooth manifold. A* continuous composition structure *on X is a smooth map*

$$F : \mathbb{R} \times X \to X, \quad (t, x) \mapsto F_t(x),$$

satisfying

1. $F_0 = \mathrm{id}_X$,

2. $F_{t+s} = F_t \circ F_s$ *for all* $t, s \in \mathbb{R}$.

The family $\{F_t\}$ defines a one-parameter group of transformations of X. No algebraic structure on X is assumed.

Infinitesimal generator

Assume F is smooth in both variables. Define the vector field

$$V(x) := \left.\frac{d}{dt}\right|_{t=0} F_t(x).$$

Proposition 2.2. *The family* F_t *satisfies the differential equation*

$$\frac{d}{dt}F_t(x) = V(F_t(x)).$$

Proof. Differentiate the identity

$$F_{t+s}(x) = F_t(F_s(x))$$

with respect to t at $t = 0$. Smoothness implies

$$\left.\frac{d}{dt}\right|_{t=0} F_{t+s}(x) = V(F_s(x)),$$

which yields the stated evolution equation. □

Thus any smooth composition law determines a vector field V, and the global family satisfies the autonomous evolution equation generated by V.

Uniqueness

Theorem 2.3 (Composition Principle)**.** *Let* F_t *be a smooth one-parameter composition structure on* X*. Then there exists a unique vector field* V *on* X *such that*

$$F_t = \exp(tV),$$

i.e., F_t *is the flow generated by* V*.*

Proof. The generator V is uniquely determined by differentiation at $t = 0$. Existence and uniqueness of flows for smooth vector fields imply that the solution to

$$\frac{d}{dt}\Phi_t(x) = V(\Phi_t(x)), \quad \Phi_0 = \mathrm{id}$$

is unique. Since F_t satisfies the same differential equation and initial condition, $F_t = \Phi_t$. □

This establishes the central structural fact: smooth associativity eliminates independent global degrees of freedom. Once smoothness and the composition law are imposed, the entire transformation family is rigidly encoded by its infinitesimal generator.

Linearization

Let x_* satisfy $F_t(x_*) = x_*$ for all t. Then $V(x_*) = 0$.

Linearizing near x_* gives

$$\frac{d}{dt}\delta x = (DV)_{x_*}\delta x.$$

Thus local classification of the composition law near a fixed point is governed entirely by the linear operator $(DV)_{x_*}$. In particular, both global behavior and local stability are controlled by first-order data.

Closure under commutator

Now suppose Φ_t and Ψ_s are two continuous symmetries with infinitesimal generators X and Y. Define the commutator transformation

$$\Theta_{t,s} = \Phi_t \circ \Psi_s \circ \Phi_t^{-1} \circ \Psi_s^{-1}.$$

Associativity of composition implies that $\Theta_{t,s}$ is well defined. Differentiating twice at $(t, s) = (0, 0)$ yields

$$[X, Y],$$

the commutator of the corresponding vector fields.

Proposition 2.4 (Inevitability of the bracket)**.** *The space of infinitesimal generators of continuous symmetries is closed under commutator. The induced operation*

$$[X, Y]$$

is bilinear, antisymmetric, and satisfies the Jacobi identity.

Proof. The commutator of vector fields is bilinear and antisymmetric by definition. The Jacobi identity follows from associativity of composition of differential operators. Since invariant flows are preserved under conjugation, the commutator of generators corresponds to the second-order variation of composed symmetries. □

We conclude that smooth associative composition not only forces infinitesimal determinism but also endows the space of generators with a Lie algebra structure. This algebra is not imposed externally; it emerges inevitably from the structural requirements of smoothness and composition.

3 Lie Groups as a Canonical Instance of the Composition Principle

We now specialize the composition principle of Section 2 to the case of smooth groups.

Lie groups as composition objects

Let G be a Lie group with smooth multiplication

$$m : G \times G \to G, \qquad (g, h) \mapsto gh,$$

identity element $e \in G$, and smooth inversion.

For each $g \in G$, define the left translation

$$L_g : G \to G, \qquad L_g(h) = gh.$$

The map $g \mapsto L_g$ is smooth, and composition satisfies

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2}.$$

Thus multiplication determines a smooth composition structure parametrized by G itself.

Infinitesimal structure at the identity

The identity element e serves as the distinguished basepoint. Define

$$\mathfrak{g} := T_e G.$$

For $v \in \mathfrak{g}$, define a vector field X_v by left translation:

$$X_v(g) := (L_g)_* v.$$

Each X_v is smooth and left-invariant. Conversely, every left-invariant vector field arises uniquely in this way. Thus

$$\mathfrak{g} \cong \{\text{left-invariant vector fields on } G\}.$$

By Section 2, the smooth composition structure on G determines infinitesimal generators. Here those generators are precisely the left-invariant fields.

Closure under commutator

The commutator of vector fields preserves left-invariance:

$$[X_v, X_w] \text{ is left-invariant.}$$

Therefore there exists a bilinear operation

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

defined by

$$[X_v, X_w] = X_{[v,w]}.$$

Theorem 3.1. *Let G be a Lie group. Then $\mathfrak{g} = T_eG$ carries a unique bilinear operation $[\ ,\ ]$ such that*

$$[X_v, X_w] = X_{[v,w]}.$$

This operation is antisymmetric and satisfies the Jacobi identity.

Proof. The commutator of vector fields is bilinear, antisymmetric, and satisfies the Jacobi identity. Since left-invariant vector fields are closed under commutator, the bracket transfers uniquely to $\mathfrak{g}$. □

Thus the Lie algebra structure is determined entirely by the smooth multiplication on G.

Functorial formulation

Let **LieGrp** denote the category of Lie groups with smooth homomorphisms, and **LieAlg** the category of Lie algebras.

Define

$$\mathrm{Lie}(G) := T_eG.$$

For a smooth homomorphism

$$f : G \to H,$$

define

$$\mathrm{Lie}(f) := (df)_e.$$

Proposition 3.2. *The assignment*

$$G \mapsto T_eG, \qquad f \mapsto (df)_e$$

defines a functor

$$\mathrm{Lie} : \mathbf{LieGrp} \to \mathbf{LieAlg}.$$

Proof. Functoriality follows from the chain rule. Since f preserves multiplication, it preserves left-invariant vector fields and therefore preserves the bracket. □

The Lie algebra is thus the infinitesimal object canonically associated to the global composition structure of the group.

4 Renormalization Flow as a Semigroup Instance of the Composition Principle

We now apply the composition principle of Section 2 to scale transformations in renormalization theory.

Scale composition structure

Let $\mathcal{T}$ denote a smooth space of theories (for example, coupling parameters or effective actions).

A renormalization procedure defines a family of maps

$$R_\ell : \mathcal{T} \to \mathcal{T}, \qquad \ell \geq 0,$$

satisfying:

1. $R_0 = \mathrm{id}_{\mathcal{T}}$,
2. $R_{\ell_1+\ell_2} = R_{\ell_1} \circ R_{\ell_2}$,
3. smooth dependence on ℓ.

Thus $\{R_\ell\}_{\ell\geq 0}$ forms a smooth one-parameter semigroup acting on $\mathcal{T}$.

This is precisely a composition structure in the sense of Section 2, with parameter restricted to $\ell \geq 0$.

Infinitesimal generator

Define the vector field

$$\beta(T) = \left.\frac{d}{d\ell}\right|_{\ell=0} R_\ell(T).$$

By the composition principle, the semigroup is determined by β through

$$\frac{d}{d\ell} R_\ell(T) = \beta(R_\ell(T)).$$

Thus the renormalization group flow is governed by the first-order differential equation

$$\frac{dT}{d\ell} = \beta(T).$$

The beta function is the infinitesimal object canonically associated to the scale composition law.

Fixed points and linearization

Let $T_* \in \mathcal{T}$ satisfy

$$R_\ell(T_*) = T_* \quad \text{for all } \ell \geq 0.$$

Then $\beta(T_*) = 0$.

Linearizing near T_* gives

$$\frac{d}{d\ell}\delta T = (D\beta)_{T_*}\, \delta T.$$

The spectrum of $(D\beta)_{T_*}$ determines the local structure of the semigroup action: directions with positive eigenvalues grow under scale, negative eigenvalues decay, and zero eigenvalues correspond to marginal directions.

Thus classification near a fixed point is determined entirely by the linearization of the generator.

Structural correspondence

The Lie group and renormalization cases now appear as parallel instances of the same structural mechanism:

Lie theory	Renormalization flow
Smooth group multiplication	Smooth scale composition
Identity element	Zero scale
Tangent at identity	Beta vector field
Lie bracket	Linearization at fixed point
Lie functor	Flow generator

In both settings, the global transformation law is encoded by a uniquely determined infinitesimal generator. The local structure is governed by the linearization of that generator.

5 Categorical Formulation and the Compression Theorem

We now formalize the structural mechanism underlying Sections 2–4.

The category of composition structures

Define a category $\mathbf{Comp}_1$ as follows:

- Objects are pairs (X, F) where X is a smooth manifold and $F : \mathbb{R}\times X \to X$ is a smooth one-parameter composition structure satisfying

$$F_0 = \mathrm{id}, \qquad F_{t+s} = F_t \circ F_s.$$

- Morphisms $(X, F) \to (Y, G)$ are smooth maps $\phi : X \to Y$ such that

$$\phi \circ F_t = G_t \circ \phi \qquad \text{for all } t \in \mathbb{R}.$$

Thus morphisms preserve the composition structure.

Let **Vect** denote the category whose objects are smooth manifolds equipped with vector fields, and whose morphisms are smooth maps that push forward one vector field to the other.

Generator assignment

To each object (X, F), associate the vector field

$$V(x) = \left.\frac{d}{dt}\right|_{t=0} F_t(x).$$

By Section 2, F_t is uniquely determined by V.

If $\phi : (X, F) \to (Y, G)$ is a morphism in $\mathbf{Comp}_1$, differentiating the relation

$$\phi(F_t(x)) = G_t(\phi(x))$$

at $t = 0$ yields

$$(d\phi)_x(V_X(x)) = V_Y(\phi(x)).$$

Thus ϕ pushes forward V_X to V_Y.

Compression theorem

Theorem 5.1 (Categorical Compression)**.** *The assignment*

$$(X, F) \mapsto (X, V), \qquad \phi \mapsto \phi,$$

defines a functor

$$\mathcal{C} : \mathbf{Comp}_1 \to \mathbf{Vect}.$$

Moreover, for each object (X, F), the global composition structure F_t is uniquely determined by its image V under this functor.

Proof. Functoriality follows from the compatibility condition for morphisms and the chain rule. Uniqueness of F_t from V is given by the Composition Principle (Section 2), which identifies F_t as the flow of V. □

Specializations

The constructions of Sections 3 and 4 are special cases of this theorem:

- A Lie group induces a composition structure via left translation, whose generator at the identity defines the Lie algebra.
- A renormalization semigroup defines a composition structure on theory space, whose generator is the beta function.

In both cases, the global transformation law is recovered from its infinitesimal generator through the functor $\mathcal{C}$.

Equivalence for complete flows

The compression theorem shows that a composition structure determines its infinitesimal generator. To make this correspondence fully symmetric, we restrict to globally defined flows.

Let $\mathbf{Comp}_1^{\mathrm{glob}}$ denote the full subcategory of $\mathbf{Comp}_1$ whose objects (X, F) satisfy that F_t is defined for all $t \in \mathbb{R}$. Let $\mathbf{Vect}^{\mathrm{comp}}$ denote the category of smooth manifolds equipped with complete vector fields.

Theorem 5.2 (Equivalence of Composition and Generator)**.** *The generator functor*

$$\mathcal{C} : \mathbf{Comp}_1^{\mathrm{glob}} \longrightarrow \mathbf{Vect}^{\mathrm{comp}}$$

is an equivalence of categories.

Proof. Given (X, F), the generator V is complete since F_t is defined for all t. Conversely, given a complete vector field V on X, its flow Φ_t exists for all t and defines a smooth composition structure. These assignments are mutually inverse on objects. Compatibility on morphisms follows from the flow–pushforward relation established in Section 5. □

6 Discussion

The results isolate a structural rigidity inherent in smooth one-parameter associative composition. Such composition laws admit a canonical compression to infinitesimal data: the global transformation family carries no independent information beyond its generator.

In Lie theory, smooth multiplication rigidifies to a Lie algebra at the identity. In renormalization theory, scale composition rigidifies to a beta vector field on theory space. Though arising in distinct domains, both constructions instantiate the same structural mechanism: smooth associativity forces infinitesimal algebra.

The categorical formulation makes this precise. The functor

$$\mathcal{C} : \mathbf{Comp}_1 \to \mathbf{Vect}$$

forgets the global family of transformations, yet the Composition Principle ensures that no information is lost. On the category of smooth one-parameter composition structures, global evolution is uniquely reconstructible from first-order data.

In this sense, smooth composition is inherently compressive. The infinitesimal generator functions as a minimal sufficient statistic of the transformation law: once specified, the entire global structure follows.

A Existence and Completeness of Flows

For completeness, we recall that if V is a smooth vector field on a manifold X, then for each point $x \in X$ there exists a maximal interval $I_x \subset \mathbb{R}$ on which the flow $\Phi_t(x)$ is defined.

A vector field is called complete if $I_x = \mathbb{R}$ for all x. On compact manifolds, every smooth vector field is complete.

This standard result justifies the restriction to complete vector fields in the equivalence theorem of Section 5.

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