Stacks and Higher Semantic Regimes

A Higher-Categorical Framework for Semantic Descent and Obstruction

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Abstract

This paper develops a higher-categorical framework for semantic structure based on the theory of stacks and cohomological obstruction. We formalize semantic regimes as locally defined interpretations over a domain, organized via descent data and compatibility conditions. While classical approaches model meaning as global invariance captured by sheaves, this framework is insufficient in settings where admissible transformations are defined only locally and higher-order coherence conditions become essential.

We extend the notion of semantic regimes from sheaves to stacks, treating local interpretations as objects, transition functions as morphisms, and higher compatibility conditions as intrinsic coherence data. Within this framework, failures of global semantic descent are classified by a hierarchy of cohomological obstructions, with first-order obstructions corresponding to nontrivial classes in H^1, and higher-order obstructions arising from the nontriviality of higher cohomology groups.

This perspective reframes meaning as a structured process of descent rather than a fixed global invariant, and establishes a direct correspondence between semantic systems and higher gauge structures. In particular, semantic disagreement and inconsistency are interpreted as curvature-like phenomena, governed by the geometry of the underlying domain and its associated cohomological invariants.

The result is a unified geometric framework in which semantic structure is intrinsically higher-dimensional, governed by descent, compatibility, and obstruction across multiple levels of coherence.

Contents

1 Introduction

The problem of semantic structure may be understood as a problem of descent: under what conditions can locally defined interpretations of observational data be assembled into a coherent global meaning?

In classical settings, meaning is identified with invariance. A quantity possesses semantic content if it remains unchanged under the admissible transformations of the system. When such transformations act globally, invariant structure is well-defined across the entire domain, and a single global semantic regime exists.

However, in more general settings—particularly those involving curvature, locality, or structural discontinuities—admissible transformations are defined only locally. In such cases, observational data admits locally consistent interpretations that may fail to extend to a global invariant structure. This failure is not incidental, but reflects an intrinsic obstruction to semantic descent.

A natural mathematical framework for organizing such local-to-global problems is sheaf theory. In this setting, semantic regimes are modeled as sheaves assigning to each domain a set of admissible interpretations, with compatibility conditions governing their agreement on overlaps. The existence of global meaning corresponds to the existence of global sections.

However, sheaves capture only first-order compatibility: they encode whether local data agrees pairwise on overlaps. They do not capture higher-order coherence conditions governing how such agreements compose across multiple overlaps. In many systems of interest, these higher-order conditions are essential.

In this paper, we show that semantic structure is fundamentally higher-categorical. We formalize semantic regimes as stacks, in which local data, transition functions, and higher coherence relations are treated as intrinsic components of a single unified object.

Within this framework:

- Local semantic interpretations correspond to objects defined over domains,
- Transition functions correspond to morphisms between local interpretations,
- Higher compatibility data correspond to coherence relations between these morphisms.

Failures of global semantic descent are thereby refined into a hierarchy of obstructions. First-order failures correspond to nontrivial classes in H^1, while higher-order failures correspond to higher cohomological invariants and the nontriviality of stack structure.

This perspective shifts the notion of meaning from invariant structure to structured coherence. Meaning is no longer identified with a single global object, but with the ability—or failure—of local structures to assemble into a coherent higher system.

As a consequence, semantic systems are not merely collections of local invariants, but higher-dimensional geometric objects governed by descent, compatibility, and obstruction. In this setting, phenomena such as horizons, discontinuities, and structural boundaries arise naturally as manifestations of nontrivial stack-theoretic structure.

The paper is organized as follows. We begin by revisiting the sheaf-theoretic formulation of semantic regimes and identifying its limitations. We then introduce stacks as the natural extension of this framework, and develop the corresponding notion of higher semantic structure. We analyze the role of higher cohomological obstructions and their interpretation in terms of semantic curvature and higher gauge theory. Finally, we provide a conceptual interpretation of semantic systems as intrinsically higher-categorical objects.

2 From Sheaves to Semantic Regimes

We begin by formalizing semantic structure using the language of sheaf theory. This provides a natural framework for organizing locally defined interpretations of observational data and their compatibility across domains.

2.1 Local Semantic Regimes

Let M be a topological space representing a domain of observation. Denote by $\mathrm{Open}(M)$ the category of open subsets of M, with morphisms given by inclusions.

We define a *semantic presheaf*

$$\mathcal{R} : \mathrm{Open}(M)^{op} \to \mathbf{Set}$$

which assigns to each open set $U \subseteq M$ a set $\mathcal{R}(U)$ of admissible semantic interpretations over U.

For each inclusion $V \subseteq U$, there is a restriction map

$$\rho_{U,V} : \mathcal{R}(U) \to \mathcal{R}(V),$$

which restricts a semantic interpretation on U to a smaller domain V.

In this formulation, elements of $\mathcal{R}(U)$ represent locally consistent meanings defined over U.

2.2 Sheaf Condition and Semantic Descent

The semantic presheaf $\mathcal{R}$ is said to be a *sheaf* if it satisfies the following gluing condition:

Let $\{U_i\}$ be an open cover of U. Suppose we are given a family of local sections

$$s_i \in \mathcal{R}(U_i)$$

such that for all i, j,

$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j).$$

Then there exists a unique global section

$$s \in \mathcal{R}(U)$$

such that

$$\rho_{U, U_i}(s) = s_i \quad foralli.$$

In this setting, semantic descent corresponds to the ability to glue locally consistent interpretations into a single global meaning.

2.3 Interpretation

The sheaf condition expresses the principle that meaning is globally coherent if and only if locally defined interpretations agree on overlaps and can be uniquely assembled into a global structure.

Thus:

- Local sections represent locally valid meanings,
- Compatibility on overlaps represents agreement between observers or domains,
- Global sections represent fully coherent semantic regimes.

When the sheaf condition is satisfied, semantic structure is well-behaved: local consistency implies global coherence.

2.4 Limitations of the Sheaf Framework

Despite its naturality, the sheaf framework captures only first-order compatibility.

Specifically, it encodes:

- Whether local interpretations agree on pairwise overlaps,
- Whether such agreement determines a global interpretation.

However, it does not encode how such agreements themselves behave across multiple overlaps.

In particular, consider triple overlaps

$$U_i \cap U_j \cap U_k.$$

Even when local sections agree pairwise, there may exist nontrivial structure in how these agreements compose. The sheaf condition does not track this higher-order compatibility.

As a result:

- Sheaves detect failure of global sections,
- But they do not capture higher-order failures of coherence.

This limitation becomes essential in systems where transition data carries intrinsic structure, such as gauge theories, cohomological systems, and the semantic frameworks developed in this work.

2.5 Motivation for Higher Structure

To fully capture semantic structure, we require a framework that encodes not only local data and pairwise compatibility, but also higher-order coherence relations governing how local agreements compose.

This leads naturally to the notion of stacks, in which:

- Local semantic regimes are treated as objects,
- Transition functions are treated as morphisms,
- Compatibility of transition functions is governed by higher coherence data.

In the next section, we introduce stacks as the natural extension of the sheaf-theoretic framework and develop a higher-categorical model of semantic regimes.

3 Stacks and Semantic Regimes

We now introduce stacks as the natural framework for modeling semantic structure beyond the limitations of sheaves.

3.1 From Sheaves to Stacks

As discussed in the previous section, sheaves encode local data together with pairwise compatibility conditions. However, they do not retain information about how these compatibilities compose across multiple overlaps.

To capture this additional structure, we must refine our notion of a semantic regime. Rather than assigning to each open set a set of interpretations, we assign a category of interpretations together with morphisms representing transformations between them.

[Semantic Fiber Category] A *semantic fiber category* over an open set $U \subseteq M$ is a category $\mathcal{R}(U)$ whose:

- objects represent admissible semantic interpretations over U,
- morphisms represent transformations between such interpretations.

Thus, semantic structure is no longer set-valued, but category-valued.

3.2 Descent Data and Gluing

Let $\{U_i\}$ be an open cover of U. A collection of local semantic data consists of:

- objects $s_i \in \mathcal{R}(U_i)$,
- isomorphisms on overlaps

$$\varphi_{ij} : s_i|_{U_i \cap U_j} \to s_j|_{U_i \cap U_j},$$

satisfying the cocycle condition on triple overlaps:

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \quad on U_i \cap U_j \cap U_k.$$

[Descent Data] Such a collection $\{s_i, \varphi_{ij}\}$ is called *descent data.*

In contrast to the sheaf case, compatibility is no longer equality of restrictions, but isomorphism mediated by transition morphisms.

3.3 Stacks

[Semantic Stack] A *semantic stack* $\mathcal{R}$ over M is a fibered category over Open(M) satisfying the following conditions:

- (Locality) Objects and morphisms are defined locally over open sets,
- (Descent) Any descent data $\{s_i, \varphi_{ij}\}$ arises from a global object $s \in \mathcal{R}(U)$,
- (Uniqueness up to isomorphism) The global object is unique up to a unique isomorphism.

Thus, stacks generalize sheaves by replacing equality with isomorphism and incorporating higher coherence conditions.

3.4 Interpretation

In this framework:

- Local semantic regimes are objects in categories $\mathcal{R}(U)$,
- Transition functions are morphisms between these objects,
- Compatibility is expressed through coherent systems of isomorphisms.

Meaning is therefore not a single invariant object, but an equivalence class within a structured system of local interpretations.

Global meaning exists when local data can be glued together coherently via isomorphisms. Failure of global meaning corresponds to the failure of such coherent descent.

3.5 Semantic Regimes as Higher Structures

The stack structure encodes multiple layers of semantic information:

- Objects represent local interpretations,
- Morphisms represent transformations between interpretations,
- Cocycle conditions encode compatibility across domains.

Thus, semantic regimes are inherently higher-dimensional structures, in which meaning is determined not only by local data, but by the structure of transformations and their coherence.

3.6 Conceptual Shift

This formulation induces a fundamental shift:

> Meaning is not an invariant object, but a descent structure in a higher category.

In particular:

- Sheaves model agreement,
- Stacks model agreement up to transformation,
- Higher structures model coherence of transformations themselves.

This perspective reveals that semantic systems are intrinsically higher-categorical objects, governed by descent, equivalence, and coherence rather than strict equality.

4 Higher Cohomological Obstructions

The stack-theoretic formulation of semantic regimes naturally leads to a hierarchy of cohomological obstructions governing the failure of global semantic descent.

4.1 Obstructions to Descent

Let $\mathcal{R}$ be a semantic stack over M, and let $\{U_i\}$ be an open cover of M.

A system of local semantic data consists of:

- objects $s_i \in \mathcal{R}(U_i)$,
- isomorphisms φ_{ij} on overlaps $U_i \cap U_j$,

satisfying compatibility conditions on triple overlaps.

When such data arises from a global object, semantic descent succeeds. When it does not, the failure is measured by cohomological invariants.

4.2 First-Order Obstruction

The first level of obstruction arises from the transition isomorphisms φ_{ij}.

These define a Čech 1-cocycle:

$$\{\varphi_{ij}\} \in Z^1(U, \mathcal{G}),$$

where $\mathcal{G}$ is the sheaf of admissible transformations.

[First-Order Semantic Obstruction] The cohomology class

$$[\{\varphi_{ij}\}] \in H^1(M, \mathcal{G})$$

measures the obstruction to the existence of a global semantic object.

When this class is nontrivial, no global semantic regime exists.

4.3 Second-Order Obstruction

Even when first-order compatibility holds, higher-order inconsistencies may arise.

On triple overlaps $U_i \cap U_j \cap U_k$, the cocycle condition requires:

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}.$$

In general, this condition may fail up to a higher transformation:

$$\varphi_{jk} \circ \varphi_{ij} = h_{ijk} \cdot \varphi_{ik},$$

where h_{ijk} represents a higher compatibility datum.

[Second-Order Semantic Obstruction] The collection $\{h_{ijk}\}$ defines a 2-cocycle whose cohomology class

$$[\{h_{ijk}\}] \in H^2(M, \mathcal{G})$$

measures the obstruction to coherent gluing of transition data.

Thus, H^2 encodes failure not of global sections, but of the consistency of the gluing process itself.

4.4 Higher Obstruction Hierarchy

This pattern extends to higher degrees.

- $H^1(M, \mathcal{G})$ measures failure of global semantic objects,
- $H^2(M, \mathcal{G})$ measures failure of coherence of transition functions,
- Higher groups $H^k(M, \mathcal{G})$ encode incompatibilities across k-fold overlaps.

[Semantic Obstruction Hierarchy] The sequence of cohomology groups

$$H^k(M, \mathcal{G}), \quad k \geq 1,$$

defines a hierarchy of semantic obstructions governing the failure of descent at increasing levels of coherence.

4.5 Stack-Theoretic Interpretation

Within the stack framework, these cohomological obstructions acquire a natural interpretation.

- Objects correspond to local semantic regimes,
- Morphisms correspond to transition functions,
- Higher morphisms correspond to coherence relations,
- Cohomology classes measure the failure of this structure to trivialize globally.

Thus, stacks provide the geometric setting in which cohomological obstructions arise naturally as intrinsic invariants of semantic structure.

4.6 Interpretation

Semantic failure is not binary, but stratified.

- First-order failure prevents global agreement,
- Second-order failure prevents consistency of agreement,
- Higher-order failures prevent coherence of the rules governing agreement.

Meaning is therefore not simply absent or present, but structured by a hierarchy of obstructions.

In this framework, semantic systems are governed by a layered geometry in which coherence conditions at each level determine the possibility of extending local interpretations into a global structure.

5 Semantic Structure and Higher Gauge Structure

The hierarchy of cohomological obstructions developed in the previous section admits a natural interpretation in terms of gauge theory and its higher generalizations.

5.1 Semantic Structure as Gauge Structure

Recall that local semantic regimes are defined over open sets $U \subseteq M$, with transition data relating interpretations across overlaps.

This structure is formally analogous to a gauge theory:

- Local semantic regimes correspond to local trivializations,
- Transition functions φ_{ij} correspond to gauge transformations,

- Global semantic regimes correspond to gauge-invariant structures.

[Semantic Gauge Structure] A *semantic gauge structure* consists of a stack $\mathcal{R}$ together with transition data encoding local equivalences between semantic regimes.

In this framework, meaning is identified with invariance under local transformations, and semantic descent corresponds to the existence of a global gauge-invariant object.

5.2 Curvature as First-Order Obstruction

In classical gauge theory, curvature measures the failure of local trivializations to be globally compatible.

Similarly, the first cohomology class

$$[\{\varphi_{ij}\}] \in H^1(M, \mathcal{G})$$

measures the failure of global semantic descent.

[Semantic Curvature] Semantic curvature is the first-order obstruction to global semantic coherence, represented by a nontrivial class in $H^1(M, \mathcal{G})$.

Thus, curvature corresponds to the impossibility of identifying a globally consistent semantic frame.

5.3 Higher Gauge Structure

The presence of higher-order obstructions requires a refinement of the gauge-theoretic framework.

In particular:

- First-order data (transition functions) correspond to gauge fields,
- Second-order data (coherence terms h_{ijk}) correspond to higher gauge fields,
- Higher cohomology classes correspond to higher curvature.

[Higher Semantic Gauge Structure] A *higher semantic gauge structure* is a stack equipped with transition data and higher coherence relations, forming a higher gauge system.

This structure generalizes classical gauge theory by incorporating transformations between transformations, and coherence conditions between these higher transformations.

5.4 Gerbes and Higher Fields

Second-order obstructions are naturally associated with gerbes.

- A gerbe encodes transition data at the level of triple overlaps,

- Its associated class in $H^2(M, \mathcal{G})$ measures higher curvature,
- It represents a higher analogue of a fiber bundle.

Thus, semantic systems with nontrivial second-order obstruction correspond to higher gauge fields rather than ordinary gauge fields.

5.5 Interpretation

Within this framework, semantic structure acquires a geometric interpretation:

- Meaning corresponds to gauge invariance,
- Local interpretations correspond to gauge choices,
- Semantic disagreement corresponds to curvature,
- Higher-order inconsistencies correspond to higher curvature.

Thus, semantic systems behave analogously to physical gauge systems, in which local data is related by transformations and global structure is constrained by curvature.

5.6 Conceptual Synthesis

This leads to the following unified picture:

Semantic regimes form a higher gauge system, whose structure is governed by cohomological invariants.

In particular:

- Sheaves correspond to flat (curvature-free) semantic systems,
- Nontrivial H^1 corresponds to curvature and failure of global meaning,
- Nontrivial H^2 corresponds to higher curvature and failure of coherent structure,
- Stacks provide the natural geometric object encoding this hierarchy.

Thus, the geometry of meaning is a gauge geometry, and its obstructions are precisely the curvature invariants of a higher gauge structure.

6 Interpretation

The preceding development establishes that semantic structure is fundamentally a problem of descent governed by higher-categorical geometry.

In classical formulations, meaning is identified with invariant structure: a quantity possesses semantic content if it remains unchanged under admissible transformations. This perspective presupposes the existence of a global regime in which such invariants are well-defined.

However, the framework developed in this paper shows that this assumption is not generally valid. In systems with localized structure, curvature, or discontinuity, meaning cannot be reduced to a single global invariant. Instead, it must be understood as arising from the coherence of locally defined interpretations.

6.1 From Invariance to Descent

The shift from sheaves to stacks reflects a deeper conceptual transition:

> Meaning is not a global invariant, but a structured process of descent.

Local semantic regimes may exist and agree up to transformation, yet fail to assemble into a globally consistent object. In such cases, meaning is not absent, but inherently local and relational.

Thus:

- Meaning is defined locally,
- Agreement is mediated by transformations,
- Global coherence depends on higher compatibility conditions.

6.2 Hierarchy of Semantic Failure

Semantic breakdown is not a single phenomenon, but a hierarchy:

- First-order failure (H^1): no global semantic regime exists,
- Second-order failure (H^2): transition structures cannot be coherently composed,
- Higher-order failure (H^k): incompatibilities arise across increasingly complex overlaps.

This hierarchy shows that semantic structure is layered, with each level governed by its own obstruction.

6.3 Geometry of Meaning

The identification of semantic regimes with stacks and their obstructions with cohomology implies that meaning possesses an intrinsic geometry.

- Local interpretations define a geometric structure over the domain,
- Transition functions define connections between these structures,
- Cohomological classes define curvature-like invariants,
- Higher coherence conditions define higher-dimensional geometric features.

Thus, semantic systems are not merely logical or representational constructs, but geometric objects governed by descent and obstruction.

6.4 Relation to Physical Systems

This geometric perspective aligns naturally with physical theories.

In gauge theory, physical quantities are defined up to local symmetry, and global structure is constrained by curvature. In higher gauge theory and string-theoretic frameworks, fields and interactions are governed by higher-order coherence conditions.

The present framework shows that semantic systems obey analogous structural principles:

- Local meaning corresponds to local gauge choice,
- Semantic equivalence corresponds to gauge transformation,
- Obstruction to global meaning corresponds to curvature,
- Higher semantic structure corresponds to higher gauge fields.

Thus, the structure of meaning and the structure of physical theory share a common geometric foundation.

6.5 Conclusion

We conclude that semantic regimes are intrinsically higher-categorical objects. Their structure is governed not by global invariance, but by descent, compatibility, and coherence across multiple levels.

Stacks provide the natural mathematical setting for this structure, and cohomology provides the language in which its obstructions are expressed.

In this framework, meaning is not a fixed object, but a geometric phenomenon: a structured system of local interpretations whose global coherence is determined by the topology and higher structure of the underlying domain.

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